1. Introduction and motivations

Supergravity (SUGRA) theories arose as attempts towards a unification of the fundamental interactions, including Quantum Gravity, and with this respect their role has been confirmed with the advent of superstring theories and, more speculatively, of the theory of supersymmetric extended objects, called super p-branes. A super p-brane lives on a $(p+1)$-dimensional world-sheet in a D-dimensional target super-space-time; the string has then to be considered as a 1-brane ($p=1$). The allowed values of D, for a given p, are dictated by classical space-time supersymmetry [1,2] and may be further restricted by consistency requirements at the quantum level. At low energies these theories can be described by SUGRA theories in D space-time dimensions and these SUGRA theories describe also the target space dynamics of the super p-brane σ-models. In the target space one can also have extended $N=2$ supersymmetry, see [3], but in this paper we concentrate on theories with simple $N=1$ space-time supersymmetry.

One of the remarkable features which arose recently in the physics of extended objects is the string ($p=1$, $D=10$) – five-brane ($p=5$, $D=10$) duality [4,5], meaning essentially that one theory can be regarded as a soliton solution of the other. According to a strong version of the duality conjecture [2,6] the two theories are equivalent in the sense that they are just different mathematical descriptions of the same underlying physics. The same should then also be true for the two corresponding $N=1$, $D=10$ supergravity theories.

In this paper we present a unified formulation of the two *pure* SUGRA theories which arise respectively as background theories of the string and five-brane σ-models at the classical level. The first SUGRA theory is usually described in terms of a closed three-superform H_3 [7] (corresponding to the string) and the second (dual) theory in terms of a closed seven-superform H_7 [8] (corresponding to the five-brane).

We discuss the issue of duality also in *non minimal* $N=1$, $D=10$ SUGRA theories which take quantum corrections to the heterotic *string* σ-model into account. In this case, in particular, the differential of H_3 is proportional to a second order polynomial in the gauge and Lorentz curvatures, $dH_3 = Tr(F^2) - tr(R^2)$, while H_7 remains closed. We leave the discussion of the SUGRA theory where the differential of H_7 becomes proportional to a fourth order polynomial in the curvatures while H_3 remains closed, which takes quantum corrections to the heterotic *five-brane* σ-model into account, to a future publication [9].

The *super*forms H_3 and H_7 are related in a way which resembles much the duality relation between three and seven-forms in *ordinary* ten-dimensional space and are therefore usually said to be "dual" to each other. The two theories, which are known to be equivalent, are most conveniently described in superspace. One has to choose an appropriate set of constraints on curvatures and torsions and then to solve the Bianchi identities. In the current treatments in the literature, according to the set of constraints one uses, one has to impose the Bianchi identity $dH_3 = 0$ to set the theory on shell [7,10] while the identity $dH_7 = 0$ does not contain any dynamical information and, in particular, does not set the theory on shell [11].

In the new formulation of $D = 10$, $N = 1$ SUGRA which we present here none of these identities are imposed as starting points, they are rather both consequences of the (simple) constraint we will impose on the super-Riemann curvature and, moreover, in this formulation the fields H_3 and H_7 are not introduced explicitly "by hand" at the beginning; the (closed) forms H_3 and H_7 will arise naturally as components of the super-curvatures and torsion and are treated in a completely symmetrical fashion: therefore in our formulation the "self-dual" nature of $D = 10$, $N = 1$ SUGRA is manifest.

The constraint on the supercurvature, mentioned above, which we introduce consists in setting to zero the spinorial components of the supercurvature two-form $R_c{}^d = \frac{1}{2} E^B E^A R_{ABc}{}^d$,

$$R_{\alpha\beta ab} = 0 \tag{1.1}$$

as suggested in [12]. Here A indicates both a vectorial index a and a spinorial index α. The constraint (1.1) resembles much the algebraic structure of the Super-Yang–Mills theory (SYM) in ten (and also in other) dimensions. We recall in fact, that if we indicate with $F = \frac{1}{2} E^A E^B F_{BA}$ the Lie algebra valued Yang–Mills curvature two-form the constraint $F_{\alpha\beta} = 0$ sets the theory on shell (in $D = 10$) in that it implies the equations of motion for gluons and gluini. As we will see, precisely the same happens also for $N=1$, $D=10$ pure SUGRA: the constraint (1.1) imposes all the equations of motion for the supergravity fields and implies, moreover, the existence of a closed three-superform and of a closed seven-superform. We would like to remember that this analogy between SUGRA and SYM holds only for pure SUGRA in that, if one constructs non minimal models e.g. coupling the supergravity to gauge fields, the constraint (1.1) can no longer be imposed [12].

A remarkable advantage of having a supercurvature two-form satisfying (1.1) results from the following considerations regarding anomalies. As is known $N=1$, $D=10$ pure SUGRA is plagued by an ABBJ Lorentz anomaly A_L due to the fact that the theory contains chiral fermions and that $D/2+1$ is even. The Lorentz anomaly A_L can be computed via standard techniques through the so-called extended transgression formula [13] starting from the twelve-form

$$X_{12} = \frac{62}{945}\ tr\ R^6 - \frac{7}{180}\ tr\ R^4\ tr\ R^2 + \frac{1}{216}\ (tr\ R^2)^3 \tag{1.2}$$

where with ${R_a}^b$ we mean here the curvature two-form in *ordinary* space. The procedure to compute A_L relies heavily on the following properties of X_{12}: it is Lorentz-invariant, closed $dX_{12}=0$, and it vanishes being a twelve-form in ten dimensions. If we indicate with Ω_L the BRST operator associated to Lorentz transformations, A_L satisfies the Wess–Zumino consistency condition:

$$\Omega_L A_L = 0. \tag{1.3}$$

It is however clear that A_L, being the standard ABBJ-anomaly, is not supersymmetric. If we indicate with Ω_S the BRST operator associated with supersymmetry $\Omega_S A_L \neq 0$, meaning that there is also a non vanishing SUSY-anomaly A_S in the theory. Therefore one has to cope with the following coupled cohomology problem [14]:

$$\begin{aligned} \Omega_L A_L &= 0 \\ \Omega_S A_L + \Omega_L A_S &= 0 \\ \Omega_S A_S &= 0. \end{aligned} \tag{1.4}$$

A straightforward extension of the transgression method, which allowed to determine A_L in ordinary space, to superspace is not available because as a *superform*

X_{12} does not vanish in $D = 10$ superspace. However, in [12] it has been shown that an explicit solution of the coupled cohomology problem (1.4) can be given provided the superform X_{12} satisfies "Weyl triviality", i.e., there exists a *Lorentz-invariant* eleven-superform Y such that

$$X_{12} = dY. \tag{1.5}$$

Note that there exists always an eleven-superform Y_{CS} of the Chern–Simons type, simply due to the fact that X_{12} is closed; $dX_{12} = 0 \;\Rightarrow\; X_{12} = dY_{CS}$, but Y_{CS} is not Lorentz-invariant. On the other hand, it can be shown [12] that Weyl triviality (1.5) holds provided the constraint $R_{\alpha\beta ab} = 0$ is satisfied.

We conclude that in the present formulation the coupled cohomology problem (1.4) can be explicitly solved and an explicit expression for the supersymmetric partner A_S of the Lorentz-anomaly can be determined, in complete analogy with the SYM theory in ten dimensions.

The issue of duality in $N = 1$, $D = 11$ SUGRA gets settled in a somehow different manner. The physical content of this theory is given by the graviton ${E_m}^a$, the gravitino ${E_m}^\alpha$ and by additional bosonic degrees of freedom which, at the kinematical level, can be described by a three-form potential B_3 or a six-form potential B_6, suggesting a duality relation between the field strengths $H_4 = dB_3$ and $\tilde{H}_7 = dB_6$.

Also in this case we reformulate the theory in superspace in a strictly supergeometrical framework without introducing any closed H_4 or $\tilde{H}_7$ at the beginning. This time the theory is put on shell by setting to zero a certain eleven-dimensional spinor superfield while in eleven dimensions $R_{\alpha\beta ab}$ remains intrinsically non vanishing in that it can not be eliminated by any field redefinitions. The Bianchi identities on the torsion imply then the existence of a 4-superform H_4 and of a 7-superform $\tilde{H}_7$ such that

$$dH_4 = 0 \tag{1.6}$$

$$d\tilde{H}_7 = 0. \tag{1.7}$$

This means that $N = 1$, $D = 11$ SUGRA is self-dual from a super-kinematical point of view, but we will see that this self-duality is broken at a dynamical level, as it is well known in the literature from many years [15]. This fact agrees of course also with the observation that there exists a super two-brane which lives in an $N = 1$, $D = 11$ SUGRA background, but that no dual p-brane, living in the same background, is known to exist.

The paper is organized as follows. In section two we discuss the general framework of our formulation of $N = 1$, $D = 10$ pure SUGRA. In section three we solve the Bianchi identities. In section four we determine the equations of motion and evidenciate the self-dual structure of the theory. Sections five and six are devoted to $N = 1$, $D = 11$ SUGRA while in section seven we discuss, in the present formulation, non minimal theories in ten and eleven dimensions. A technical appendix containing our conventions and some group-theoretical considerations on $SO(10)$ and $SO(11)$ concludes the paper.

2. The structure of pure supergravity in ten dimensions

The $N = 1$, $D = 10$ pure supergravity [16] multiplet is given by the graviton $E_m{}^a$, the chiral gravitino $E_m{}^\alpha$, the dilaton ϕ, a chiral fermion which we call gravitello V_α and by additional 28 bosonic degrees of freedom which can be described either by a 2-form potential $B_{a_1a_2}$ or by a 6-form potential $B_{a_1-a_6}$ [17].

A superspace in ten dimensions [7] is spanned by the coordinates $z^M = (x^m, \theta^\mu)$ where x^m $(m = 0, 1, \ldots, 9)$ are the ordinary space-time coordinates and θ^μ $(\mu = 1, \ldots, 16)$ are Grassmann variables. We introduce the supervielbein one-forms $E^A = dz^M E_M{}^A(z)$ where $A = \{a, \alpha\}$ $(a = 0, 1 \ldots, 9;\ \alpha = 1, \ldots, 16)$ is a flat index (letters from the beginning of the alphabet represent flat indices: small latin letters indicate vectorial indices, small greek letters indicate spinorial indices and capital letters denote both of them). The p-superforms can be decomposed in the vielbein basis as

$$\phi_p = \frac{1}{p!} E^{A_1} - E^{A_p} \phi_{A_p - A_1}(z).$$

We denote the Lorentz-valued super- spin connection one-form by $\Omega_A{}^B = dz^M \Omega_{MA}{}^B =$

$E^C\Omega_{CA}{}^B$ and the corresponding covariant differential is written as D, while d indicates the ordinary superspace differential. A superfield $\psi_A{}^B$ is said to be Lorentz-valued if $\psi_{ab} = -\psi_{ba}$ and $\psi_\alpha{}^\beta = \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta\psi_{ab}$. Here we defined

$$\Gamma^{a_1 - a_k} \equiv \Gamma^{[a_1} - \Gamma^{a_k]}$$

and the matrices $(\Gamma^a)_{\alpha\beta}$ and $(\Gamma_a)^{\alpha\beta}$ are Weyl matrices satisfying the Weyl algebra (see the appendix)

$$(\Gamma^a)_{\alpha\beta}(\Gamma^b)^{\beta\gamma} + (\Gamma^b)_{\alpha\beta}(\Gamma^a)^{\beta\gamma} = 2\eta^{ab}\delta_\alpha^\gamma.$$

The torsion two-form and the Lorentz-valued curvature two-form are defined respectively as

$$\begin{aligned} T^A &= DE^A = \frac{1}{2}E^B E^C T_{CB}{}^A \\ R_A{}^B &= d\Omega_A{}^B + \Omega_A{}^C\Omega_C{}^B = \frac{1}{2}E^C E^D R_{DCA}{}^B \end{aligned} \tag{2.1}$$

and satisfy the Bianchi identities

$$DT^A = E^B R_B{}^A \tag{2.2}$$

$$DR_A{}^B = 0. \tag{2.3}$$

Notice that we do not introduce any two- or six-form potential.

The above introduced superfields contain a huge number of unphysical fields which have to be eliminated by imposing suitable constraints on the torsion $T_{AB}{}^C$ and on the curvature $R_{ABC}{}^D$. Once constraints are imposed the Bianchi identities are no longer identities and they have to be solved consistently.

As it has been shown in [18] once the torsion Bianchi identities (2.2) are consistently solved the Bianchi identities for the curvature (2.3) are automatically satisfied. This implies that it is sufficient to solve the torsion Bianchi identities which in components read as:

$$D_{[A}T_{BC)}{}^D + T_{[AB}{}^G T_{GC)}{}^D = R_{[ABC)}{}^D \tag{2.4}$$

where the symbol $[\cdots)$ indicates graded symmetrization. In what follows $[\cdots]$ will denote antisymmetrization and $(\cdots)$ symmetrization of indices.

Our starting point is the fundamental rigid supersymmetry preserving constraint

$$T_{\alpha\beta}{}^{a} = 2\Gamma^{a}_{\alpha\beta}. \tag{2.5}$$

Starting from this constraint we can simplify the other components of the torsion via field redefinitions through the following considerations. In terms of irreducible representations (irrep) of $SO(10)$ we can decompose $T_{\alpha\beta}{}^{\gamma}$ and $T_{\alpha a}{}^{b}$ as follows:

$$T_{\alpha\beta}{}^{\gamma} = (1440 \oplus 560 \oplus 144 \oplus 2 \cdot 16) \tag{2.6}$$

$$T_{\alpha a}{}^{b} = (720 \oplus 560 \oplus 2 \cdot 144 \oplus 2 \cdot 16) \tag{2.7}$$

Through the field redefinitions [19,20]

$$\begin{aligned} E'^{\alpha} &= E^{\alpha} + E^{b}\, H_{b}{}^{\alpha} \\ \Omega'_{\alpha a}{}^{b} &= \Omega_{\alpha a}{}^{b} + X_{\alpha a}{}^{b}, \end{aligned} \tag{2.8}$$

where $H_{b}{}^{\alpha}$ and $X_{\alpha a}{}^{b}$ are suitable covariant superfields, we can eliminate from $T_{\alpha a}{}^{b}$ all the irreps apart from the 720. Writing now (2.4) in the lowest sector

$$(\Gamma^{a})_{\delta(\alpha} T_{\beta\gamma)}{}^{\delta} = (\Gamma^{g})_{(\alpha\beta} T_{\gamma)g}{}^{a} \tag{2.9}$$

and noting that according to (2.6) $T_{\alpha\beta}{}^{\gamma}$ does not contain the irrep 720 also $T_{\alpha a}{}^{b}$ can not contain it and must therefore vanish. Noting that the general content of irreps in (2.9) is $5280 \oplus 1440 \oplus 720 \oplus 560 \oplus 144 \oplus 16$ and that the r.h.s. of (2.9) is zero, we conclude that $T_{\alpha\beta}{}^{\gamma}$ can contain only an irrep 16, which corresponds to a spinor V_{α}. A short calculation gives then

$$T_{\alpha\beta}{}^{\gamma} = 2\delta^{\gamma}_{(\alpha} V_{\beta)} - (\Gamma^{g})_{\alpha\beta} (\Gamma_{g})^{\gamma\varphi} V_{\varphi}. \tag{2.10}$$

All these considerations were of purely kinematical nature.

We introduce now dynamics by imposing that the purely spinorial components of the supercurvature vanish:

$$R_{\alpha\beta ab} = 0\,; \tag{2.11}$$

we will in fact see that with this constraint the theory is set on shell.

To summarize, our basic parametrizations for supercurvature and torsion are

$$\begin{aligned} T_{\alpha\beta}{}^{a} &= 2\Gamma^{a}_{\alpha\beta} \\ T_{\alpha\beta}{}^{\gamma} &= 2\delta^{\gamma}_{(\alpha}V_{\beta)} - (\Gamma^{g})_{\alpha\beta}(\Gamma_{g})^{\gamma\varphi}V_{\varphi} \\ T_{\alpha a}{}^{b} &= 0 = T_{a\alpha}{}^{b} \\ R_{\alpha\beta ab} &= 0. \end{aligned} \tag{2.12}$$

In the next section we will see that the closure of the superalgebra implies a constraint on the superfield V_{α} which can be identically solved if one says that V_{α} is the spinorial derivative of a scalar superfield, the dilaton ϕ

$$V_{\alpha} = D_{\alpha}\phi. \tag{2.13}$$

We would like to stress that, without any additional assumption, (2.12) and (2.13) are sufficient to determine the theory completely and to imply in particular all the equations of motion demanding the closure of the SUSY-algebra, as will be seen in the next section. Under "closure of the SUSY-algebra" we understand the consistency of the Bianchi identities with the commutator algebra:

$$D_{A}D_{B} - (-)^{AB}D_{B}D_{A} = -T_{AB}{}^{C}D_{C} - R_{AB\#}{}^{\#}. \tag{2.14}$$

3. Solution of the torsion Bianchi identities

The Bianchi identities (2.4) which have to be solved are written more explicitly as follows (the one with the lowest dimensions, see (2.9), has already been solved):

$$2T_{a(\alpha}{}^{\gamma}(\Gamma^{b})_{\beta)\gamma} + (\Gamma^{g})_{\alpha\beta}T_{ga}{}^{b} = 0 \tag{3.1}$$

$$D_{(\alpha}T_{\beta\gamma)}{}^{\delta} - 2(\Gamma^{g})_{(\alpha\beta}T_{\gamma)g}{}^{\delta} + T_{(\alpha\beta}{}^{\varepsilon}T_{\gamma)\varepsilon}{}^{\delta} = 0 \tag{3.2}$$

$$D_{\alpha}T_{ab}{}^{c} + 2(\Gamma^{c})_{\alpha\gamma}T_{ab}{}^{\gamma} = 2R_{\alpha[ab]}{}^{c} \tag{3.3}$$

$$D_a T_{\alpha\beta}{}^{\gamma} + 2D_{(\alpha}T_{\beta)a}{}^{\gamma} + 2T_{a(\alpha}{}^{\delta}T_{\beta)\delta}{}^{\gamma} + 2\Gamma^g_{\alpha\beta}T_{ga}{}^{\gamma} - T_{\alpha\beta}{}^{\varepsilon}T_{a\varepsilon}{}^{\gamma} = 2R_{a(\alpha\beta)}{}^{\gamma} \quad (3.4)$$

$$D_{[a}T_{bc]}{}^{d} - T_{[ab}{}^{g}T_{c]g}{}^{d} = R_{[abc]}{}^{d} \quad (3.5)$$

$$D_\alpha T_{ab}{}^{\beta} + 2D_{[a}T_{b]\alpha}{}^{\beta} - 2T_{\alpha[a}{}^{\delta}T_{b]\delta}{}^{\beta} + T_{ab}{}^{g}T_{g\alpha}{}^{\beta} + T_{ab}{}^{\delta}T_{\delta\alpha}{}^{\beta} = R_{ab\alpha}{}^{\beta} \quad (3.6)$$

we want now to present the (unique) solution of (3.1)-(3.6) in compatibility with (2.12) and (2.13).

The solution of these equations can be most easily achieved using group theoretical considerations: every tensor appearing in the equations gets first decomposed in irreps of $SO(10)$, then the general content of irreps of each equation has to be established and finally the equations are solved in each sector of $SO(10)$ irreps separately; this procedure reduces the necessary Γ-matrix gymnastic to a minimum.

Eqs. (3.1) and (3.2) are solved as follows: they imply that the vectorial torsion $T_{ab}{}^{c}$ is completely antisymmetric in its three indices and corresponds thus to a 120 irrep of $SO(10)$:

$$T_{abc} \equiv T_{ab}{}^{d}\eta_{dc} = T_{[abc]}; \quad (3.7)$$

moreover,

$$T_{a\alpha}{}^{\beta} = \frac{1}{4}(\Gamma^{bc})_{\alpha}{}^{\beta}T_{abc} \quad (3.8)$$

$$D_\alpha D_\beta \phi = -\Gamma^g_{\alpha\beta}D_g\phi + V_\alpha V_\beta + \frac{1}{12}(\Gamma_{abc})_{\alpha\beta}T^{abc} \quad (3.9)$$

(remember that $V_\alpha = D_\alpha\phi$). These relations represent the unique solution to (3.1) and (3.2).

Eqs. (3.3) and (3.4) are solved by the following relations:

$$D_\alpha T_{abc} = -6T_{[ab}{}^{\varepsilon}(\Gamma_{c]})_{\varepsilon\alpha} \quad (3.10)$$

$$R_{a\alpha bc} = 2(\Gamma_a)_{\alpha\varepsilon} T^{\varepsilon}_{bc} \tag{3.11}$$

$$(\Gamma^b)_{\alpha\varepsilon} T_{ba}{}^{\varepsilon} = D_a V_\alpha + \frac{1}{4} T_{abc} (\Gamma^{bc})_\alpha{}^\beta V_\beta. \tag{3.12}$$

Eq. (3.11) says that the curvature with one vector- and one spinor-like index is proportional to the field strength of the gravitino, $T_{ab}{}^{\alpha}$, in analogy with the Yang–Mills case [10] where one gets

$$F_{a\alpha} = (\Gamma_a)_{\alpha\varepsilon} \chi^{\varepsilon}$$

where χ^{ε} is the gluino superfield. (3.12) is the equation of motion for the gravitino.

Eq. (3.5) is the purely vectorial Bianchi identity for a curvature with torsion and has thus not to be "solved". It implies in particular that the antisymmetric part of the Ricci tensor $R_{ab} \equiv R_{acb}{}^{c}$ is non vanishing,

$$R_{[ab]} = -\frac{1}{2} D^c T_{cab}. \tag{3.13}$$

Eq. (3.6) has to be regarded as an equation which determines the spinorial derivative of $T_{ab}{}^{\beta}$, i.e. the supersymmetry transformations law for the gravitino field strength.

In the next section we will enforce the closure of the SUSY-algebra via (2.14) to derive the equations of motion and to prove the self-dual character of the theory.

4. The equations of motion: duality as an outcome

So far we have only obtained the equation of motion for the gravitino, (3.12). The equation of motion for the graviton can be obtained contracting (3.6) with $(\Gamma^a \Gamma_c)_\beta{}^\alpha$ and using in the first term on the r.h.s. the gravitino equation (3.12). One gets:

$$R_{bc} = 2 D_c D_b \phi - D_a T^a{}_{bc}, \tag{4.1}$$

and symmetrizing this one obtains Einstein's equations

$$R_{(ab)} = 2D_{(a}D_{b)}\phi. \tag{4.2}$$

To obtain the equation of motion for the gravitello V_α one has to work a little bit harder. In the conventional formulations, see for example [7][10][21], this equation is obtained demanding the existence of a closed three-superform, suitably constrained. In the present case we can obtain it by imposing the closure of the SUSY-algebra on (3.9). We compute

$$D_{(\varepsilon}D_{\alpha)}D_\beta\phi = -\Gamma^g_{\varepsilon\alpha}D_gD_\beta\phi - \frac{1}{2}T_{\varepsilon\alpha}{}^\delta D_\delta V_\beta \tag{4.3}$$

and equate this expression to the one obtained applying D_ε to (3.9). The net result we get is the equation of motion for the gravitello:

$$(\Gamma^a)^{\alpha\beta}D_aV_\beta = 2(\Gamma^a)^{\alpha\beta}V_\beta D_a\phi - \frac{1}{12}(\Gamma_{abc})^{\alpha\beta}T^{abc}V_\beta. \tag{4.4}$$

The equation of motion for the dilaton follows now as usual by applying D_α to this equation:

$$D^aD_a\phi = 2D_a\phi D^a\phi - \frac{1}{12}T_{abc}T^{abc}. \tag{4.5}$$

The reconstruction of H_3

Now we come to the "reconstruction" of the equations for the gravi-photon. We want first construct a closed three-form. To do this we compute

$$D_{(\beta}D_{\alpha)}T_{abc} = -\Gamma^g_{\alpha\beta}D_gT_{abc} - \frac{1}{2}T_{\alpha\beta}{}^\delta D_\delta T_{abc} \tag{4.6}$$

and equate this expression to the one obtained applying D_β to (3.10) and using on the r.h.s. again (3.6). One gets, upon projecting with $(\Gamma_d)^{\alpha\beta}$,

$$D_dT_{abc} - 6D_{[a}T_{bc]d} + 3R_{[abc]d} - 9T^2_{[abc]d} = 0 \tag{4.7}$$

where we defined $T^2_{abcd} \equiv T_{ab}{}^gT_{gcd}$. Comparing this with the Bianchi identity (3.5) we obtain precisely what we need:

$$D_{[a}T_{bcd]} + \frac{3}{2}\, T^2_{[abcd]} = 0 \tag{4.8}$$

In fact, if we define now a three-superform $H_3 = \frac{1}{3!}E^A E^B E^C H_{CBA}$ through:

$$\begin{aligned} H_{\alpha\beta\gamma} &= 0 = H_{ab\alpha} \\ H_{a\alpha\beta} &= 2\,(\Gamma_a)_{\alpha\beta} \\ H_{abc} &= T_{abc} \end{aligned} \tag{4.9}$$

the relations (3.10), (4.8) and the cyclic identity $(\Gamma^a)_{(\alpha\beta}(\Gamma_a)_{\gamma)\delta} = 0$, imply then that

$$dH_3 = 0 \tag{4.10}$$

and therefore

$$H_3 = dB_2 \tag{4.11}$$

for some two-superform B_2 (throughout this paper we assume that there are no topological obstructions and so all closed forms are also exact). We conclude that we can interpret T_{abc} as the curl of a two-form potential, and its equation of motion can be read off from (3.13) and (4.1):

$$D_c T^c{}_{ab} = -4 D_{[a} D_{b]}\phi. \tag{4.12}$$

The reconstruction of H_7

It is a little bit less straightforward to construct a closed seven-superform starting from (4.12). We proceed through the following steps. First we observe that we can rewrite (4.12) as

$$D_c\Big(e^{-2\phi} T^c{}_{ab}\Big) = 2e^{-2\phi} T_{ab}{}^\alpha V_\alpha. \tag{4.13}$$

Defining now a 120 irrep as $V_{abc} = (\Gamma_{abc})^{\alpha\beta} V_\alpha V_\beta$ we can use the gravitino and gravitello equations of motion to obtain its divergence as

$$D^c\Big(e^{-2\phi}V_{abc}\Big) = e^{-2\phi}\Big((\Gamma^{hg}{}_{ab})_\varepsilon{}^\beta T_{hg}{}^\varepsilon V_\beta + 2T_{ab}{}^\alpha V_\alpha + T^{c_1c_2}{}_{[a}V_{b]c_1c_2}\Big). \tag{4.14}$$

Eliminating now the term $T_{ab}{}^\alpha V_\alpha$ between (4.13) and (4.14) we get

$$D^c\Big(e^{-2\phi}(T_{abc} - V_{abc})\Big) = -e^{-2\phi}\Big((\Gamma^{hg}{}_{ab})_\varepsilon{}^\beta T_{hg}{}^\varepsilon V_\beta + T^{c_1c_2}{}_{[a}V_{b]c_1c_2}\Big). \tag{4.15}$$

With the definitions

$$\begin{aligned} H_{a_1-a_7} &\equiv \frac{1}{3!}\,\varepsilon_{a_1-a_7b_1b_2b_3}e^{-2\phi}\Big(T^{b_1b_2b_3} - V^{b_1b_2b_3}\Big) \\ H_{\alpha a_1-a_6} &\equiv -2e^{-2\phi}(\Gamma_{a_1-a_6})_\alpha{}^\beta V_\beta \end{aligned} \tag{4.16}$$

(4.15) can be recast into

$$D_{[a_1}H_{a_2-a_8]} + \frac{7}{2}\,T_{[a_1a_2}{}^b H_{a_3-a_8]b} + \frac{7}{2}\,T_{[a_1a_2}{}^\delta H_{\delta a_3-a_8]} = 0. \tag{4.17}$$

If we now define, moreover,

$$\begin{aligned} H_{\alpha\beta a_1-a_5} &\equiv -2e^{-2\phi}(\Gamma_{a_1-a_5})_{\alpha\beta} \\ H_{\alpha_1\alpha_2\alpha_3a_1-a_4} &= \cdots = H_{\alpha_1\alpha_2\ldots\alpha_7} = 0 \end{aligned} \tag{4.18}$$

it is a simple (but lengthy) exercise to show that the seven-superform

$$H_7 \equiv \frac{1}{7!}E^{A_1} - E^{A_7}H_{A_7-A_1} \tag{4.19}$$

satisfies identically

$$dH_7 = 0. \tag{4.20}$$

Equation (4.17) is clearly the projection of (4.20) on the purely vectorial sector. Thus we proved the existence of a six-superform B_6 such that

$$H_7 = dB_6.$$

The theory admits therefore a double interpretation: we can regard T_{abc} as a closed three-superform whose equation of motion is given by (4.12); otherwise we can express T_{abc} through the first equation of (4.16), in terms of a closed seven-form $H_{a_1-a_7}$ whose equation of motion can be read off directly from (4.8):

$$
\begin{aligned}
D_b H^b{}_{a_1-a_6} &= -2D_g\phi\ H^g{}_{a_1-a_6} - 6V_{b_1b_2[a_1}H^{b_1b_2}{}_{a_2-a_6]} \\
&+\frac{1}{3!}\varepsilon_{a_1-a_6}{}^{b_1-b_4}e^{-2\phi}\left(D_{b_1}V_{b_2-b_4}+\frac{3}{2}\ V^2_{b_1b_2b_3b_4}\right) \\
&-\frac{1}{180}e^{2\phi}\varepsilon_{a_1-a_6b_1-b_4}H^{b_1b_2}{}_{f_1-f_5}H^{b_3b_4f_1-f_5}
\end{aligned}
\tag{4.21}
$$

where $V^2_{abcd} = V_{ab}{}^g V_{gcd}$.

It is worthwhile to notice that (4.8) and (4.21) simplify naturally if one introduces the torsion-free covariant derivative, $\tilde{D}_a$, which is defined in terms of the torsion-free connection

$$
\tilde{\Omega}_{ab}{}^c = \Omega_{ab}{}^c - \frac{1}{2}T_{ab}{}^c. \tag{4.22}
$$

(4.8) and (4.21) become then simply:

$$
\begin{aligned}
\tilde{D}_{[a}T_{bcd]} = \tilde{D}_{[a}H_{bcd]} &= 0 \\
\tilde{D}_g(e^{2\phi}H^g{}_{a_1-a_6}) &= \frac{1}{3!}\varepsilon_{a_1-a_6}{}^{b_1-b_4}\tilde{D}_{b_1}V_{b_2b_3b_4}.
\end{aligned}
$$

Notice, however, that the shift (4.22) would introduce a non vanishing $R_{\alpha\beta ab}$, as it can easily be seen, and therefore we did not perform this shift.

It is important to notice that the fundamental duality relation (4.16) involves only tensors which are invariant under the gauge transformations

$$
B_6 \to B_6 + d\phi_5
$$

$$
B_2 \to B_2 + d\phi_1
$$

where $\phi_{1,5}$ are arbitrary superforms (contrary to what happens in $N=1$, $D=11$ SUGRA, as we will see). This implies definitely that (self)-duality holds also at the dynamical level, meaning that one can write a gauge invariant action and gauge invariant equations of motion in which appears only H_7, or a gauge invariant action and gauge invariant equations of motion in which appears only H_3.

The analysis of the self-duality property in non minimal $D=10$ supergravity theories will be developed in section seven.

5. N=1, D=11 Supergravity: a group theoretical analysis of the constraints

We want derive in the following two sections the superspace equations of motion of $N = 1$, $D = 11$ SUGRA according to a strategy analogous to the one used in the preceding sections to evidenciate the self-duality nature of $N = 1$, $D = 10$ pure SUGRA. Through an exhaustive group theoretical analysis of possible constraints we want also to show in which direction one has to move if one wants to construct non minimal $N = 1$, $D = 11$ supergravity theories. Such non minimal theories are interesting in that they can take quantum corrections to the classical super two-brane σ-model into account, supposed that the super two-brane is a consistent theory also at the quantum level.

We take here the conservative point of view demanding that the zero-dimension component of the torsion is the rigid one:

$$T_{\alpha\beta}{}^{a} = 2\Gamma^{a}_{\alpha\beta}$$

(for conventions about Γ-matrices and notations, see the appendix). Our starting points are again the Bianchi identities (2.2) and (2.3), Dragon's theorem holds also here and we have thus to find a consistent solution of (2.4).

We remember that the $N = 1$, $D = 11$ SUGRA multiplet is made out of the graviton, $E_m{}^a$, the gravitino $E_m{}^\alpha$ and additional bosonic degrees of freedom which "numerically" can be described in terms of a three-form B_3 or a six-form B_6 potential. Also here we do not introduce any closed four- or seven-superform a priori, but try to reconstruct them in *superspace* by solving solely (2.4).

To begin with, we apply the same kinematics as in section two. The decompositions in terms of $SO(11)$ irreps, analogous to (2.6) and (2.7), are now

$$T_{\alpha\beta}{}^{\gamma} = 5280 \oplus 4224 \oplus 3520 \oplus 2 \cdot 1408 \oplus 3 \cdot 320 \oplus 3 \cdot 32 \tag{5.1}$$

$$T_{\alpha a}{}^{b} = 1760 \oplus 1408 \oplus 2 \cdot 320 \oplus 2 \cdot 32. \tag{5.2}$$

Precisely as in section two through the field redefinitions (2.8) we can now eliminate from $T_{\alpha a}{}^b$ all irreps apart from the 1760. The lowest order Bianchi identity is formally identical to (2.9):

$$(\Gamma^a)_{\delta(\alpha} T_{\beta\gamma)}{}^{\delta} = (\Gamma^g)_{(\alpha\beta} T_{\gamma)g}{}^{a}. \tag{5.3}$$

Again the 1760 irrep is not contained in $T_{\alpha\beta}{}^\gamma$ and so (5.3) implies that $T_{\alpha a}{}^b$ vanishes. Moreover, the general content of (5.3) is given by

$$36960 \oplus 10240 \oplus 5280 \oplus 4224 \oplus 3520 \oplus 1760 \oplus 2 \cdot 1408 \oplus 3 \cdot 320 \oplus 2 \cdot 32;$$

taking a look at (5.1) and noticing that the r.h.s. of (5.3) vanishes, we conclude that all irreps of $T_{\alpha\beta}{}^{\gamma}$ have to vanish, apart from *one* 32 (the spinorial representation). This is due to the fact that $T_{\alpha\beta}{}^{\gamma}$ contains three 32 irreps and that (5.3) establishes two linear relations among them; therefore only one of them is independent. A short computation gives then

$$T_{\alpha\beta}{}^{\gamma} = 16\delta^{\gamma}_{(\alpha}V_{\beta)} - 6(\Gamma^{g})_{\alpha\beta}(\Gamma_{g})^{\gamma\delta}V_{\delta} + (\Gamma^{ab})_{\alpha\beta}(\Gamma_{ab})^{\gamma\delta}V_{\delta} \tag{5.4}$$

where we identify V_{α} as the independent 32 irrep. This relation substitutes eq. (2.10) in ten dimensions. A more fundamental difference between $D = 10$ and $D = 11$ SUGRA comes in at this point: the theory is set on shell and reduces to *pure* $N = 1$, $D = 11$ SUGRA if we set (as a dynamical constraint):

$$T_{\alpha\beta}{}^{\gamma} = 0 \ \Leftrightarrow \ V_{\alpha} = 0. \tag{5.5}$$

As we will see $R_{\alpha\beta ab}$ is in this case *intrinsically* different from zero, it can not be set to zero by field redefinitions. So with respect to the ten-dimensional case the situation is completely reversed: there we could set $R_{\alpha\beta ab} = 0$ and then $T_{\alpha\beta}{}^{\gamma}$ survived, here it is precisely the opposite!

Once we have established that $R_{\alpha\beta ab}$ is different from zero we can shift the vectorial connection $\Omega_{ab}{}^{c}$ to set $T_{ab}{}^{c}$ to zero (notice that in $D = 10$ the analogous shift in the connection was not performed in that it would have turned on $R_{\alpha\beta ab}$).

To conclude, $D = 11$, $N = 1$ pure supergravity can be derived through the following constraints

$$\begin{aligned} T_{\alpha\beta}{}^{a} &= 2\Gamma^{a}_{\alpha\beta} \\ T_{\alpha a}{}^{b} &= T_{\alpha\beta}{}^{\gamma} = T_{ab}{}^{c} = 0. \end{aligned} \tag{5.6}$$

What we learned mainly from all this is that, within the framework in which $T_{\alpha\beta}{}^{a} = 2\Gamma^{a}_{\alpha\beta}$, the *unique* way to construct a non minimal $N = 1$, $D = 11$ SUGRA (modulo field redefinitions) is to introduce a non vanishing spinorial 32 irrep in $T_{\alpha\beta}{}^{\gamma}$. We will comment on the possible significances of this relaxed constraint and on the importance the resulting theories would have in section seven. Here we proceed by rederiving minimal $N = 1$, $D = 11$ SUGRA relying on group theoretical reasonings and paying particular attention to the duality structure of the theory.

The Bianchi identities are very similar to (3.1)-(3.6). Eqs. (3.1) and (3.2) are substituted by:

$$
\begin{aligned}
&4T_{a(\alpha}{}^{\gamma}\Gamma^{b}_{\beta)\gamma} = R_{\alpha\beta a}{}^{b} \\
&2\Gamma^{g}_{(\alpha\beta}T_{g\gamma)}{}^{\delta} = R_{(\alpha\beta\gamma)}{}^{\delta}
\end{aligned}
\tag{5.7}
$$

(remember that $R_{\alpha\beta\gamma}{}^{\delta} = \frac{1}{4}R_{\alpha\beta ab}\Gamma^{ab}{}_{\gamma}{}^{\delta}$). The remaining Bianchi identities are obtained from (3.3)-(3.6) by simply enforcing (5.6) and we will not write them down explicitly.

The group theoretical reasoning which allows one to solve (5.7) is reported in the appendix. Here we state simply the result.

It turns out that $R_{\alpha\beta ab}$ and $T_{a\alpha}{}^{\beta}$ are expressed in terms of *one* 330 irrep (which corresponds to a completely antisymmetric rank four tensor) W_{abcd}. One gets

$$
\begin{aligned}
T_{a\alpha}{}^{\beta} &= 8\ (\Gamma^{b_1b_2b_3})_{\alpha\beta}W_{ab_1b_2b_3} + (\Gamma_{ab_1-b_4})_{\alpha\beta}W^{b_1-b_4} \\
R_{\alpha\beta ab} &= 96\ (\Gamma^{c_1c_2})_{\alpha\beta}W_{c_1c_2ab} + 4\ (\Gamma_{abc_1-c_4})_{\alpha\beta}W^{c_1-c_4}.
\end{aligned}
\tag{5.8}
$$

Equations (3.3) and (3.4) are solved by the relations

$$
R_{\alpha abc} = 2T_{a[b}{}^{\delta}\Gamma_{c]\delta\alpha} - T_{bc}{}^{\delta}(\Gamma_{a})_{\delta\alpha} \tag{5.9}
$$

$$
D_{\alpha}W_{a_1-a_4} = \frac{1}{24}(\Gamma_{[a_1a_2})_{\alpha\gamma}T_{a_3a_4]}{}^{\gamma}. \tag{5.10}
$$

Consistency implies also that among the three irreps contained in $T_{ab}{}^{\alpha}$, i.e. $1408\oplus 320\oplus 32$, only the highest one, i.e. the 1408, is non vanishing. This implies immediately the gravitino equation of motion

$$
(\Gamma^{abc})_{\alpha\beta}T_{bc}{}^{\beta} = 0 \tag{5.11}
$$

and that the "trace" of $T_{ab}{}^{\alpha}$ vanishes:

$$
T_{ab}{}^{\alpha}(\Gamma^{b})_{\alpha\beta} = 0. \tag{5.12}
$$

6. Duality in D=11

The identity (3.5), remembering that now $T_{ab}{}^{c} = 0$, implies that $R_{c[ab]}{}^{c} = 0$ meaning that the Ricci tensor is symmetric, $R_{ab} = R_{(ab)}$. (3.6) instead can be written as

$$D_\alpha T_{ab}{}^\beta = -2D_{[a}T_{b]\alpha}{}^\beta - 2T_{[a\alpha}{}^\gamma T_{b]\gamma}{}^\beta + \frac{1}{4}R_{abcd}(\Gamma^{cd})_\alpha{}^\beta. \tag{6.1}$$

Using then the tracelessness of $T_{ab}{}^\beta$ and contracting (6.1) with $(\Gamma^b\Gamma_c)_\beta{}^\alpha$ we obtain Einstein's equations

$$R_{ab} - \frac{1}{2}\eta_{ab}R = -288\cdot 4!\,\left(W^2_{ab} - \frac{1}{8}\,\eta_{ab}W^2\right) \tag{6.2}$$

where we defined $R = R^a{}_a$, $W^2_{ab} \equiv W_{ac_1c_2c_3}W_b{}^{c_1c_2c_3}$, $W^2 \equiv W_{a_1-a_4}W^{a_1-a_4}$.

We compute now

$$(\Gamma_{a_5})^{\beta\alpha}D_\beta D_\alpha W_{a_1-a_4} = -32D_{a_5}W_{a_1-a_4}. \tag{6.3}$$

The left hand side of this equation can also be evaluated by applying D_β to (5.10) and using eq. (6.1) for $D_\beta T_{a_3a_4}{}^\gamma$.

The net result of this computation is the important relation

$$D_{[a_1}W_{a_2a_3a_4a_5]} = 0. \tag{6.4}$$

If we define now

$$\begin{aligned} H_{a_1-a_4} &= W_{a_1-a_4} \\ H_{ab\alpha\beta} &= -\frac{1}{144}(\Gamma_{ab})_{\alpha\beta} \\ H_{\alpha\beta\gamma\delta} &= H_{\alpha\beta\gamma a} = H_{\alpha abc} = 0 \end{aligned} \tag{6.5}$$

and then as usual $H_4 = \frac{1}{4!}E^{A_1} - E^{A_4}H_{A_4-A_1}$, eq. (6.4), together with other relations of the present and the preceding section imply that H_4 is a closed superform

$$dH_4 = 0 \tag{6.6}$$

and therefore we can define a 3-form superpotential B_3 such that

$$H_4 = dB_3. \tag{6.7}$$

Using now again the tracelessness of the gravitino field strength in the form

$$D_\alpha T_{ag}{}^\beta(\Gamma^g\Gamma_{bc})_\beta{}^\alpha = 0$$

and substituting (6.1) we get

$$D_d W^d{}_{abc} = -\frac{1}{4}\varepsilon_{abcf_1-f_4g_1-g_4} W^{f_1-f_4} W^{g_1-g_4} \tag{6.8}$$

which can be read as the equation of motion for H_4. However, if we define a seven-superform $H_7 = \frac{1}{7!}E^{A_1} - E^{A_7} H_{A_7-A_1}$ through

$$\begin{aligned} H_{a_1-a_7} &= \frac{1}{4!}\varepsilon_{a_1-a_7b_1-b_4} W^{b_1-b_4} \\ H_{\alpha\beta a_1-a_5} &= \frac{1}{144}\ (\Gamma_{a_1-a_5})_{\alpha\beta} \end{aligned} \tag{6.9}$$

and all other components of $H_{A_1-A_7}$ vanishing, then (6.8), together with other relations of the last two sections, implies the *superspace* relation

$$dH_7 = \frac{1}{144} H_4 \wedge H_4. \tag{6.10}$$

Substituting (6.7) we can write this as

$$d\left(H_7 - \frac{1}{144}B_3 \wedge H_4\right) = 0$$

meaning that the seven-superform $\tilde{H}_7 \equiv H_7 - \frac{1}{144}B_3 \wedge H_4$ is closed,

$$d\tilde{H}_7 = 0 \qquad \Rightarrow \qquad \tilde{H}_7 = dB_6, \tag{6.11}$$

for some six-superform B_6. (6.7) and (6.11) imply that in $N = 1$, $D = 11$ SUGRA duality holds at the *kinematical* level in superspace, meaning that one can always construct a closed four-superform and a closed seven-superform. We observe also that we can write

$$H_7 = dB_6 + \frac{1}{144}B_3 \wedge H_4 \tag{6.12}$$

which resembles much the relation which couples in $N = 1$, $D = 10$ the super-Maxwell multiplet to $N = 1$, $D = 10$ SUGRA [22]:

$$H_3 = dB_2 + kA \wedge F \tag{6.13}$$

where A and F are the connection 1-form and curvature 2-form respectively. Gauge invariance in (6.13), $A \to A + d\phi$, is saved by demanding that

$$B_2 \to B_2 - k\phi \wedge F.$$

Similarly we can save gauge invariance in (6.12) by demanding that $B_3 \to B_3 + d\phi_2$ be accompanied by

$$B_6 \to B_6 - \frac{1}{144}\phi_2 \wedge H_4.$$

Thus gauge invariance and duality hold, at a kinematical level, in superspace.

From a dynamical point of view, however, one has to observe that if one reads the eqs. of motion (6.2) and (6.8) in terms of H_4, then the potential B_3 appears obviously in a local way simply because $H_4 = dB_3$; on the other hand those equations can also be interpreted as equations of motion which involve H_7 through local (polynomial) expressions, see (6.9), and the equation of motion for H_7 would then simply be (see (6.4))

$$D^b H_{ba_1-a_6} = 0. \tag{6.14}$$

However, the relation between B_6 and H_7 becomes now *non local*. In fact, if one "inverts" the relation $H_4 = dB_3$ to get a non local expression for B_3 in terms of H_4, or equivalently in terms of H_7, (6.12) produces an implicit and non local relation between H_7 and B_6. We conclude, therefore, that in the dual interpretation, i.e. in terms of a closed seven-form, $N = 1$, $D = 11$ SUGRA becomes non local, as it is already known in the literature of course, and we are forced to formulate the theory in terms of a closed four-form.

7. Conclusions and further developments

Let us first make some remarks on supergravity theories in ten dimensions.

As we saw, in our approach in the pure supergravity theory a closed three-superform and a closed seven-superform arise naturally, the unique dynamical constraint being $R_{\alpha\beta ab} = 0$. For, to couple the theory to e.g. Yang–Mills fields or to construct non minimal couplings in pure supergravity theories (or both) one has to release this constraint introducing a non vanishing $R_{\alpha\beta ab}$. On completely general grounds relying only on the constraint $T_{\alpha\beta}{}^a = 2\,\Gamma^a_{\alpha\beta}$, one can find that the most general parametrization of $R_{\alpha\beta ab}$, modulo field redefinitions, is in terms of a single 120 irrep superfield [19]

$$R_{\alpha\beta ab} = (\Gamma_{abc_1c_2c_3})_{\alpha\beta} J^{c_1c_2c_3} \tag{7.1}$$

where the 120 irrep J^{abc} plays the role of a current. Accordingly, $T_{a\alpha}{}^{\beta}$ gets corrected to

$$T_{a\alpha}{}^{\beta} = \frac{1}{4}(\Gamma^{bc})_{\alpha}{}^{\beta} T_{abc} - \frac{1}{4}(\Gamma_{abcd})_{\alpha}{}^{\beta} J^{bcd}. \tag{7.2}$$

The solution of the torsion-Bianchi identities with the (most general) parametrization (7.1) leads to a modification of all equations of motion and to one constraint on the highest irrep contained in the spinorial derivative of J_{abc}

$$\left[D_{\alpha}(e^{2\phi} J_{abc})\right]^{1200} = 0. \tag{7.3}$$

Once this constraint is satisfied it can be shown that, starting from J_{abc}, one can construct a closed four-superform K, such that

$$dH_3 = K \tag{7.4}$$

where H_3 is again defined as in (4.9) and

$$\begin{aligned} K_{\alpha\beta\gamma\delta} &= K_{\alpha\beta\gamma a} = 0 \\ K_{\alpha\beta ab} &= 2(\Gamma_{abc_1c_2c_3})_{\alpha\beta} J^{c_1c_2c_3} \\ dK &= 0 \end{aligned} \tag{7.5}$$

with some more complicated expressions for $K_{\alpha abc}$ and K_{abcd}. Therefore a three-superform Ω exists such that $K = d\Omega$ and hence $d(H_3 - \Omega) = 0$, or

$$H_3 = dB_2 + \Omega \tag{7.6}$$

for some two-form potential B_2.

Similarly one can show that it is also possible to construct a closed seven-superform H_7

$$dH_7 = 0 \tag{7.7}$$

through

$$\begin{aligned} H_{a_1-a_7} &= \frac{1}{3!}\, e^{-2\phi} \varepsilon_{a_1-a_7}{}^{b_1-b_3}(T_{b_1-b_3} - V_{b_1-b_3} - 6J_{b_1-b_3}) \\ H_{\alpha a_1-a_6} &= -2e^{-2\phi}(\Gamma_{a_1-a_6})_{\alpha}{}^{\varepsilon} V_{\varepsilon} \\ H_{\alpha\beta a_1-a_5} &= -2e^{-2\phi}(\Gamma_{a_1-a_5})_{\alpha\beta}, \end{aligned} \tag{7.8}$$

while all other components of H_7 are vanishing. Remember that, according to (4.9), which holds also in the extended case under investigation in the present section, $T_{abc} = H_{abc}$.

We conclude that also in the case of $D = 10$ extended SUGRA models the theory can be read in two ways: either (7.7) is interpreted as the Bianchi identity for the B_6 potential, and then (7.4) is its equation of motion, or (7.4) is interpreted as the Bianchi identity for B_2 through (7.6), and then (7.7) is its equation of motion.

Clearly all this fits precisely in what is known in the literature. In fact, in order to couple to the SYM fields, one can search for a decomposition of the type

$$d\omega_{YM} = TrF^2 = dX_{YM} + K_{YM} \tag{7.9}$$

where $F = \frac{1}{2}E^A E^B F_{BA}$ is the Lie algebra valued Yang–Mills supercurvature two-form and ω_{YM} is the associated Chern–Simons three-superform. Choosing for F standard constraints, i.e. $F_{\alpha\beta} = 0$, it can easily be shown that (7.9) holds, with a four-form K_{YM} satisfying (7.5), if one chooses for X_{YM} the gauge-invariant 3-superform

$$X_{YM} = -\frac{1}{48} E^c E^b E^a (\Gamma_{abc})_{\alpha\beta} Tr(\chi^\alpha \chi^\beta).$$

Here χ^α is the gluino superfield (a Lie algebra valued spinor).

The coupling to the Lorentz Chern–Simons form is formally analogous; however, now it is not trivial to show that [21][23][24]

$$d\omega_L = trR^2 \equiv {R_a}^b {R_b}^a = dX_L + K_L,$$

where X_L is a Lorentz invariant 3-form, whose explicit expression is rather lengthy and can be found e.g. in [21,24], and K_L satisfies again (7.5). In both cases, as it is well known, B_2 has to transform anomalously under gauge transformations because ω_L and ω_{YM} are not invariant. The invariant superforms X_{YM} and X_L redefine simply H_3, in that Ω is given by $\Omega = (\omega_{YM} - \omega_L) - (X_{YM} - X_L)$ (see below).

There is a third case of relevance in the literature [25,26] in which K is the differential of an *invariant* three-superform, called Z in [25,26], which gives rise to superstring corrections of the minimal pure supergravity which are not dictated by anomaly cancellation like, for example, to a term in the action which is proportional to the fourth power of the Riemann curvature, $(R_{abcd})^4$.

Our main conclusion with respect to $N=1$, $D=10$ supergravity theories is that, on completely general grounds, relying on the unique (kinematical) hypothesis that the zero-dimension component of the torsion be flat, $T_{\alpha\beta}{}^a = 2\,\Gamma^a_{\alpha\beta}$, there exist always closed three and seven-superforms, and that the theory is therefore intrinsically self-dual; to repeat, this is true for *every* non minimal extension of the theory based on $T_{\alpha\beta}{}^a = 2\,\Gamma^a_{\alpha\beta}$.

Finally, we would like to recall that, as we saw, for non minimal theories $R_{\alpha\beta ab}$ is no longer zero (see (7.1)) and therefore it is far from obvious that the twelve superform which triggers the ABBJ Lorentz, gauge and mixed anomalies [27] satisfies the Weyl triviality property (1.5) (we remember that this property allows one to compute the supersymmetric partner of the ABBJ anomaly). We hope to be able to prove Weyl triviality for this twelve superform in our formulation in that, in contrast to previous formulations, if one switches off the "external current" J_{abc} the extended models reduce to a Weyl trivial model (pure SUGRA) thanks to $R_{\alpha\beta ab}=0$; this constraint is a sufficient condition for Weyl triviality to hold, but it should not be necessary.

The three above mentioned non-minimal extensions of the theory are relevant in that the resulting extended SUGRA theories describe the low-energy dynamics of the heterotic superstring. Particularly interesting is the theory based on the 3-form field strength

$$\begin{aligned} H_3 &= dB_2 + (\omega_{YM} - X_{YM}) - (\omega_L - X_L) \\ dH_3 &= TrF^2 - trR^2 - d(X_{YM} - X_L) \end{aligned} \tag{7.10}$$

which is related to the Green–Schwarz anomaly cancellation mechanism [27]. Recently it has been argued that the heterotic five-brane whose background theory is an $N=1$, $D=10$ SUGRA, is dual to the heterotic string [4] and, as a test of this conjecture, it has been shown [5] that the Lorentz and gauge anomalies cancel in the heterotic five-brane via a mechanism which can be regarded as dual to the Green–Schwarz one. It is based on a seven-form H_7 satisfying (in ordinary "bosonic" space)

$$
\begin{aligned}
dH_7 = & \frac{1}{24}TrF^4 - \frac{1}{7200}(TrF^2)^2 - \frac{1}{240}TrF^2trR^2 + \frac{1}{8}trR^4 + \frac{1}{32}(trR^2)^2 \\
& \equiv X_8 = d\omega_7
\end{aligned}
\tag{7.11}
$$

where ω_7 is a generalized Chern–Simons form. As originally the Green–Schwarz mechanism, also the "dual" mechanism [28] based on (7.11) breaks supersymmetry. To restore supersymmetry one had to find a consistent solution of the Bianchi identity (7.11) written in superspace. We hope that our formulation of $N = 1$, $D = 10$ SUGRA, which gave us a new general insight into the self-dual nature of the theory, permits us to answer definitely the question of the compatibility of (7.11) with supersymmetry. This issue is of some importance because, if a consistent heterotic five-brane exists, then a consistent $N = 1$, $D = 10$ supergravity, based on (7.11), must also exist and in this case one would have (formally) a new theory, the five-brane, describing the same physics as the heterotic string. Interesting applications of this equivalence could result for example from the observation that duality interchanges classical with quantum corrections and therefore, instead of making a quantum computation in string theory, one could perform a classical computation in the five-brane.
We will discuss the consistency of (7.11) with supersymmetry elsewhere [9].

Regarding $N = 1$, $D = 11$ SUGRA we would like to comment briefly on the possible extensions of the minimal theory based on (5.4) with a non vanishing spinor V_α. It is clear that this spinor has not to be a new field, but must be a (covariant) function of the fields already present in the theory, and clearly V_α would have to satisfy a certain number of constraints coming from the Bianchi identities. If the extended theory has to be consistent then the torsion Bianchi identities have to imply the existence of a closed 4-superform (otherwise the gauge invariance, needed to eliminate the unphysical degrees of freedom of the graviphoton, is missing). The issue of existence of such extended $N = 1$, $D = 11$ supergravity theories is of some relevance because the *classical* supermembrane ($p = 2$) lives in an $N = 1$, $D = 11$ minimal supergravity background and the fundamental k-invariance of the supermembrane σ-model holds true classically if the background fields satisfy the equations of motion of minimal SUGRA. If the σ-model is consistent also at the quantum level [29] then one can compute the k-anomalies and the requirement of their cancellation could then give rise to local non minimal supergravity theories. Along these lines proposals for non minimal SUGRA theories have been made in [29] via a cohomological analysis of k-anomalies in the supermembrane σ-model. It would be interesting to find out if our general framework for non minimal $N = 1$, $D = 11$ SUGRA, based only on the rigid SUSY preserving constraint $T_{\alpha\beta}{}^{a} = 2\ \Gamma^{a}_{\alpha\beta}$, fits with the extensions proposed in [29]; this check, based on a detailed analysis of non minimal models, will be the subject of a future publication [30].

Acknowledgements

The authors would like to thank Mario Tonin for numerous invaluable discussions.

Appendix

1. Ten dimensional gamma-matrix algebra

We use a Majorana–Weyl representation for the Dirac matrices γ^a(*)

$$
\begin{aligned}
(\Gamma^a)_{\alpha\beta} &= (\gamma^a)_\alpha{}^{\underline{\varepsilon}} C_{\underline{\varepsilon}\beta} \\
(\Gamma^a)^{\alpha\beta} &= C^{\alpha\underline{\varepsilon}} (\gamma^a)_{\underline{\varepsilon}}{}^\alpha,
\end{aligned}
\tag{A.1}
$$

where C is the (antisymmetric and idempotent) charge conjugation matrix, characterized by the Weyl algebra

$$
(\Gamma^a)_{\alpha\beta}(\Gamma^b)^{\beta\gamma} + (\Gamma^b)_{\alpha\beta}(\Gamma^a)^{\beta\gamma} = 2\eta^{ab}\delta_\alpha{}^\gamma \tag{A.2}
$$

(here η^{ab} is the "mostly minus" metric). We define

$$
\Gamma^{a_1\cdots a_k} = \Gamma^{[a_1} \cdots \Gamma^{a_k]} \tag{A.3}
$$

that are subjected to the duality property

$$
\Gamma^{a_1\cdots a_k} = \pm(-1)^{\frac{k}{2}(k+1)} \frac{1}{(D-k)!} \varepsilon^{a_1\cdots a_D} \Gamma_{a_{k+1}\cdots a_D} \tag{A.4}
$$

with the minus sign when the first matrix has low spinor indices, and the plus sign in the other case. Characteristic of ten dimensions is the cyclic identity

$$
(\Gamma^g)_{(\alpha\beta}(\Gamma_g)_{\gamma)\delta} = 0 \tag{A.5}
$$

which implies its "dual"

$$
(\Gamma^g)_{(\alpha\beta}(\Gamma_g{}^{a_1\cdots a_4})_{\gamma\delta)} = 0. \tag{A.6}
$$

(*) The $\underline{\varepsilon}$ index runs over the full 32 components of Dirac spinors, while the other indices are in the chiral 16 (lower indices) or $\overline{16}$ (upper indices) irrep of $SO(10)$.

2. Eleven dimensional gamma matrix algebra

In eleven dimensions we switch to the "mostly plus" metric to avoid the appearance of explicit "i" factors in the formalism. Through the charge conjugation matrix C we define the matrices

$$\begin{aligned}
(\Gamma^a)_\alpha{}^\beta &= (\gamma^a)_\alpha{}^\beta \\
(\Gamma^a)_{\alpha\beta} &= (\gamma^a)_\alpha{}^\varepsilon C_{\varepsilon\beta} \\
(\Gamma^a)^{\alpha\beta} &= C^{\alpha\varepsilon}(\gamma^a)_\varepsilon{}^\beta \\
(\Gamma^a)^\alpha{}_\beta &= C^{\alpha\varepsilon}(\gamma^a)_\varepsilon{}^\lambda C_{\lambda\beta}.
\end{aligned} \tag{A.7}$$

The symmetric matrices are

$$\Gamma^a, \Gamma^{a_1a_2}, \Gamma^{a_1\cdots a_5}, \Gamma^{a_1\cdots a_6}, \Gamma^{a_1\cdots a_9}, \Gamma^{a_1\cdots a_{10}}$$

while the antisymmetric ones are

$$C, \Gamma^{a_1\cdots a_3}, \Gamma^{a_1\cdots a_4}, \Gamma^{a_1\cdots a_7}, \Gamma^{a_1\cdots a_8}, \Gamma^{a_1\cdots a_{11}}.$$

The duality property becomes

$$\Gamma^{a_1\cdots a_k} = -(-1)^{\frac{k}{2}(k-1)}\frac{1}{(D-k)!}\varepsilon^{a_1\cdots a_D}\Gamma_{a_{k+1}\cdots a_D}; \tag{A.8}$$

the cyclic identity reads

$$(\Gamma^{ga})_{(\alpha\beta}(\Gamma_g)_{\gamma\delta)} = 0 \tag{A.9}$$

and its "dual" is now

$$(\Gamma^g)_{(\alpha\beta}(\Gamma_g{}^{a_1\cdots a_4})_{\gamma\delta)} = 3(\Gamma^{[a_1a_2})_{(\alpha\beta}(\Gamma^{a_3a_4]})_{\gamma\delta)}, \tag{A.10}$$

which signals the non vanishing of the curl of H_7.

3. Solution of the dimension one Bianchi identities in D=11

Here the dimension one Bianchi identities are:

$$4T_{a(\alpha}{}^\varepsilon(\Gamma_b)_{\beta)\varepsilon} = R_{\alpha\beta ab} \tag{A.11}$$

$$2(\Gamma^g)_{(\alpha\beta}T_{g\gamma)}{}^\delta = R_{(\alpha\beta\gamma)}{}^\delta. \tag{A.12}$$

The irrep content of $T_{a\alpha\beta} = T_{a\alpha}{}^{\gamma} C_{\gamma\beta}$ is:

$$\begin{aligned} T_{a\alpha\beta} &= T_{a(\alpha\beta)} + T_{a[\alpha\beta]} \\ &= (11 \oplus 55 \oplus 462) \otimes 11 \oplus (1 \oplus 165 \oplus 330) \otimes 11 \\ &= 4290 \oplus 3003 \oplus 1430 \oplus 2 \cdot 462 \oplus 429 \\ &\quad \oplus 2 \cdot 330 \oplus 2 \cdot 165 \oplus 65 \oplus 2 \cdot 55 \oplus 2 \cdot 11 \oplus 1. \end{aligned} \tag{A.13}$$

Symmetrizing (A.11) in (ab) and taking into account that $R_{\alpha\beta ab}$ is antisymmetric in a, b we get

$$T^{(a}{}_{(\alpha}{}^{\varepsilon} \Gamma^{b)}_{\beta)\varepsilon} = 0. \tag{A.14}$$

The general irrep content of (A.14) is given by

$$\begin{aligned} (ab)(\alpha\beta) &= (1 \oplus 65) \otimes (11 \oplus 55 \oplus 462) \\ &= 22275 \oplus 4290 \oplus 3003 \oplus 2025 \oplus 1430 \oplus 2 \cdot 462 \\ &\quad \oplus 429 \oplus 275 \oplus 65 \oplus 2 \cdot 55 \oplus 2 \cdot 11 \end{aligned} \tag{A.15}$$

so that

$$T_{a\alpha}{}^{\beta} = 2 \cdot 330 \oplus 2 \cdot 165 \oplus 1. \tag{A.16}$$

Due to (A.11) $R_{\alpha\beta a}{}^{b}$ contains at most the irreps contained in (A.16). Now we can combine eqs. (A.11) and (A.12). Eq. (A.12) contains (among a large set of irreps which we are not interested in) the irreps $3 \cdot 330 \oplus 3 \cdot 165 \oplus 1$. By direct inspection one finds that the three linear equations in (A.12) involving the two 165 irreps are linearly independent and therefore the two 165 have to vanish, the equation on the singlet implies its vanishing while the three equations on the two 330 are found to be linearly dependent from only one of them, meaning simply that the two 330 have to be proportional to each other. We conclude that $T_{a\alpha}{}^{\beta}$ and $R_{\alpha\beta a}{}^{b}$ are made out of a single 330 irrep (a fourth rank antisymmetric tensor W_{abcd}) in two different forms as shown in (5.8).

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DFPD/93/TH/51
hep-th/9309143

DUALITY IN SUPERGRAVITY THEORIES*

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Abstract

The target space dynamics of supermembrane (and superstring) theories is described by supergravity theories. Supergravity theories associated to dual supermembrane theories live in the same space-time dimension and are themselves dual to each other. We present a unified treatment in superspace of the two dual formulations of $D = 10$, $N = 1$ *pure* supergravity based on a strictly supergeometrical framework: the only fundamental objects are the super Riemann curvature and torsion, and the related Bianchi identities are sufficient to set the theory on shell; there is no need to introduce, from the beginning, closed three- or seven-superforms. This formulation extends also to *non minimal* models. Moreover, in this framework the algebraic analogy between pure super Yang–Mills theories and pure supergravity in $D = 10$ is manifest. As an additional outcome in the present formulation the supersymmetric partner of the ABBJ-Lorentz anomaly in pure $D = 10$ supergravity can be computed in complete analogy to the ABBJ-gauge anomaly in supersymmetric Yang–Mills theories in ten dimensions. In the same framework we attack the issue of duality in $N = 1$, $D = 11$ supergravity showing in detail that duality holds at the kinematical level in superspace while it is broken by the dynamics. We discuss also possible extensions of this theory which could be related to quantum corrections of the eleven dimensional membrane.

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* Supported in part by M.P.I. This work is carried out in the framework of the European Community Programme "Gauge Theories, Applied Supersymmetry and Quantum Gravity" with a financial contribution under contract SC1-CT92 -D789.