

The $N = 1$, $D = 10$ SUGRA–SYM theory is known to be plagued by an ABBJ gauge and Lorentz anomaly which, for the gauge groups $SO(32)$ or $E_8 \times E_8$, cancels via the Green–Schwarz anomaly cancellation mechanism; the harmlessness of this anomaly is reflected also in the corresponding superstring theories [1]. It is, however, immediately seen that the appearance of this gauge–Lorentz anomaly \mathcal{A}_G implies also the appearance of a (local) supersymmetry anomaly \mathcal{A}_S . If we call Γ the effective action of the SUGRA–SYM theory and $\Omega_S(\Omega_G)$ the nilpotent BRST operator associated to supersymmetry transformations (gauge and Lorentz transformations) we have, apart from $o(\hbar^2)$ terms,

$$\Omega_G \Gamma = \mathcal{A}_G \quad (1)$$

where \mathcal{A}_G is the usual ABBJ gauge–Lorentz anomaly [2,3]. This anomaly is clearly not invariant under supersymmetry transformations and so

$$0 \neq \Omega_S \mathcal{A}_G = \Omega_S \Omega_G \Gamma = -\Omega_G(\Omega_S \Gamma) \quad (2)$$

meaning that

$$\Omega_S \Gamma = \mathcal{A}_S \neq 0, \quad (3)$$

i.e. Γ suffers also a supersymmetry anomaly. Eqs.(1)-(3) define the following coupled cohomology problem

$$\begin{aligned} \Omega_G \mathcal{A}_G &= 0 \\ \Omega_G \mathcal{A}_S + \Omega_S \mathcal{A}_G &= 0 \\ \Omega_S \mathcal{A}_S &= 0 \end{aligned} \quad (4)$$

which, for the known ABBJ anomaly \mathcal{A}_G , can be considered as a system of equations for the unknown supersymmetry anomaly \mathcal{A}_S , the “supersymmetric partner” of \mathcal{A}_G . An analogous cohomology problem can be defined also for supersymmetric chiral Yang–Mills theories in $D = 4$ and 6 dimensions. Explicit solutions for \mathcal{A}_S satisfying (4) have been found for gauge anomalies in $D = 4, 6$ [4] in the case of global and local supersymmetry and for the Lorentz anomaly in pure supergravity theories in $D = 6, 10$ [5] due to a remarkable property called “Weil triviality” (see below) in those references. However, in the case of the coupled SUGRA–SYM theory in $D = 10$, i.e. the theory under investigation in this letter, it is difficult to prove Weil triviality and a general solution of (4) has not yet been found.

As is well known, for the gauge groups $SO(32)$ and $E_8 \times E_8$, there exists a local counterterm Δ (see below) which trivializes the cocycle \mathcal{A}_G

$$\mathcal{A}_G = \Omega_G \Delta. \quad (5)$$

The information contained in (5), together with the cohomology eqs. (4), generally speaking, do not allow one to conclude that also the supersymmetry anomaly trivializes. In this letter we want a) solve (4) for these particular gauge groups giving an explicit expression for \mathcal{A}_S , b) show that \mathcal{A}_S is a trivial cocycle of Ω_S and c) show that \mathcal{A}_S gets trivialized by the same Δ appearing in (5).

$$\mathcal{A}_S = \Omega_S \Delta. \quad (6)$$

We would like to notice that, among all cases solved in the literature until now, only in the case of the SYM theory with global SUSY in $D = 4$ dimensions \mathcal{A}_S is known to be a trivial cocycle of Ω_S [4] (in that case, this holds true even if \mathcal{A}_G is non trivial). To summarize our result: the Green–Schwarz counterterm cancels also the supersymmetry anomaly.

Our notation is as follows. We work in a ten-dimensional superspace spanned by the coordinates $z^M = (x^m, \vartheta^\mu)$ where x^m ($m = 0, 1, \dots, 9$) are the ordinary space–time coordinates and ϑ^μ ($\mu = 1, \dots, 16$) are Grassmann variables. We introduce the supervielbein one-forms $E^A = dz^M E_M^A(z)$ where $A = \{a, \alpha\}$ ($a = 0, 1, \dots, 9$; $\alpha = 1, \dots, 16$) is a flat index (letters from the beginning of the alphabet represent flat indices, letters from the middle of the alphabet onwards represent curved indices: small latin letters indicate vectorial indices, small greek letters indicate spinorial indices and capital letters denote both of them). p -superforms can be decomposed along the vielbein basis or along the coordinate basis,

$$\phi = \frac{1}{p!} E^{A_1} \dots E^{A_p} \phi_{A_p \dots A_1} = \frac{1}{p!} dz^{M_1} \dots dz^{M_p} \phi_{M_p \dots M_1}. \quad (7)$$

We denote the Lorentz-valued super spin connection one-form by $\omega_a^b = dz^M \omega_{M a}^b = E^C \omega_{C a}^b$ ($\omega_{ab} = -\omega_{ba}$) and the corresponding Lorentz–curvature two-superform by $R_a^b = d\omega_a^b + \omega_a^c \omega_c^b = \frac{1}{2} E^D E^C R_{C D a}^b$. d indicates the differential in superspace. We introduce a Yang–Mills connection one-superform $A = E^B A_B$ with values in the Lie algebra of a group H ($SO(32)$ or $E_8 \times E_8$ in the following) and the

Lie-algebra valued curvature two-superform $F = dA + AA = \frac{1}{2}E^B E^C F_{CB}$. To construct the relevant nilpotent BRST operators we introduce the ghost fields $\xi^M(z)$ for superdiffeomorphisms, a Lie algebra valued ghost field $C(z)$ for gauge transformations and an $SO(10)$ Lorentz-valued ghost field $u_a{}^b(z)$ for local Lorentz transformations. Local supersymmetry transformations are represented by the superdiffeomorphisms. The BRST transformations are the following:

Superdiffeomorphisms:

$$\begin{aligned}
\delta_S \xi^M &= \xi^L \partial_L \xi^M \\
\delta_S C &= \xi^L \partial_L C \\
\delta_S u_a{}^b &= \xi^L \partial_L u_a{}^b \\
\delta_S \phi &= L_\xi \phi \equiv (d i_\xi - i_\xi d) \phi
\end{aligned} \tag{8}$$

Gauge transformations:

$$\begin{aligned}
\delta_H C &= -CC \\
\delta_H A &= -dC - AC - CA \\
\delta_H u_a{}^b &= \delta_H \xi^M = \delta_H \omega_a{}^b = 0
\end{aligned} \tag{9}$$

Local Lorentz transformations:

$$\begin{aligned}
\delta_L u_a{}^b &= -u_a{}^c u_c{}^b \\
\delta_L \omega_a{}^b &= -du_a{}^b - \omega_a{}^c u_c{}^b - u_a{}^c \omega_c{}^b \\
\delta_L C &= \delta_L \xi^M = \delta_L A = 0 \\
\delta_L V_{A_1 \dots A_n} &= -u_{A_1}{}^B V_{BA_2 \dots A_n} - (\text{permutations})
\end{aligned} \tag{10}$$

L_ξ indicates the Lie derivative along the vector field $\xi = \xi^M \partial_M$, the interior product with a superform ϕ is defined as $i_\xi \phi = \frac{1}{(p-1)!} \xi^{M_1} dz^{M_2} \dots dz^{M_p} \phi_{M_p \dots M_1}$, $u_\alpha{}^\beta = \frac{1}{4}(\Gamma_{ab})_\alpha{}^\beta u^{ab}$, $u_a{}^\alpha = u_\alpha{}^a = 0$, and $V_{A_1 \dots A_n}$ in (10) indicates a Lorentz-tensor. We note the useful relation

$$\delta_S(i_\xi \phi) = \frac{1}{2}(d i_\xi i_\xi - i_\xi i_\xi d) \phi \tag{11}$$

where ϕ is a superform with zero ghost number.

The superfields (or forms) of the theory constitute a graded algebra if we define the grading n_ψ of a p -form ψ as $n_\psi = p + g_\psi$, where g_ψ is the ghost number

of ψ , and associate a grading 1 to the operators d and δ . d and δ begin to operate on a composite term from the right. There are the usual additional signs if the fields carry spinorial indices [6].

It is convenient to combine the transformations in (9) and (10) to define BRST transformations associated to the “total” gauge group $G = H \otimes SO(10)$ through

$$\delta_G \equiv \delta_H + \delta_L. \quad (12)$$

Then we can associate to δ_S and δ_G the corresponding BRST operators Ω_S and Ω_G , satisfying: $\Omega_G^2 = \Omega_S\Omega_G + \Omega_G\Omega_S = \Omega_S^2 = 0$. These operators define then the coupled cohomology problem (4). We try to solve it, for $H = SO(32)$ or $E_8 \times E_8$, according to a method developed in [4,5] which represents a generalization of the method for the determination of ABBJ anomalies, based on the extended transgression formula (ETF) [7], to the supersymmetric case.

For the above gauge groups the relevant sixth-order polynomial in the curvatures factorizes, as is well known [3], into

$$I_{12} = X_4 X_8 \quad (13)$$

with

$$\begin{aligned} X_4 &= \frac{1}{30} Tr F^2 - tr R^2 \\ X_8 &= \frac{1}{24} Tr F^4 - \frac{1}{7200} (Tr F^2)^2 - \frac{1}{240} Tr F^2 tr R^2 + \frac{1}{8} tr R^4 + \frac{1}{32} (tr R^2)^2 \end{aligned} \quad (14)$$

(Tr denotes the trace in the adjoint representation, tr denotes the trace in the fundamental representation). We denote collectively $\mathcal{F} = (F, R)$, $\mathcal{A} = (A, \omega)$, $\mathcal{C} = (C, u_a{}^b)$. X_4 and X_8 are closed G -invariant superforms. For the moment we pursue formally the ETF method. Define:

$$\hat{d} = d + \delta_G \quad (15)$$

$$\hat{\mathcal{A}} = \mathcal{A} + \mathcal{C} \quad (16)$$

$$\hat{\mathcal{F}} = \hat{d}\hat{\mathcal{A}} + \hat{\mathcal{A}}\hat{\mathcal{A}} = \mathcal{F} \quad (17)$$

$$\hat{\mathcal{F}}_t = t(\hat{d}\hat{\mathcal{A}} + t\hat{\mathcal{A}}\hat{\mathcal{A}}) \quad (18)$$

where the real parameter t runs from 0 to 1. Then one has the identities

$$\begin{aligned} X_4 &= \hat{d} \left(2 \int_0^1 dt X_4(\hat{\mathcal{A}}, \hat{\mathcal{F}}_t) \right) \equiv \hat{d} Q_3 \\ X_8 &= \hat{d} \left(4 \int_0^1 dt X_8(\hat{\mathcal{A}}, \hat{\mathcal{F}}_t) \right) \equiv \hat{d} Q_7. \end{aligned} \tag{19}$$

With the integrands in (19) we mean the expressions which are obtained from X_4 and X_8 by substituting *one* of the curvatures with $\hat{\mathcal{A}}$ and the remaining ones with $\hat{\mathcal{F}}_t$ and symmetrizing then with unit weight (the explicit expressions of Q_3 and Q_7 are, however, not relevant for our purposes). Q_3 and Q_7 can be decomposed in sectors with different ghost number p

$$\begin{aligned} Q_3 &= \sum_{p=0}^3 X_{p,3-p} \\ Q_7 &= \sum_{p=0}^7 X_{p,7-p} \end{aligned} \tag{20}$$

and (19) implies then the following identities (descent equations)

$$\begin{aligned} dX_{0,3} &= X_4 \\ dX_{1,2} + \delta_G X_{0,3} &= 0 \\ dX_{2,1} + \delta_G X_{1,2} &= 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned} dX_{0,7} &= X_8 \\ dX_{1,6} + \delta_G X_{0,7} &= 0 \\ dX_{2,5} + \delta_G X_{1,6} &= 0. \end{aligned} \tag{22}$$

Moreover, due to the fact that X_4 and X_8 are G-invariant and closed we have

$$\begin{aligned} I_{12} &= X_4 X_8 = \frac{2}{3} X_4 \hat{d} Q_7 + \frac{1}{3} \hat{d} Q_3 X_8 \\ &= \hat{d} \left(\frac{2}{3} X_4 Q_7 + \frac{1}{3} Q_3 X_8 \right). \end{aligned} \tag{23}$$

If we were now in ordinary space the left hand side in (23) would vanish, being a twelve form in ten dimensions, and the first non trivial descent equation associated to (23) would identify the ABBJ G-anomaly. In superspace, however, I_{12} is non vanishing. This problem can be bypassed if, in superspace, I_{12} can be written

as the differential of a G -invariant eleven-superform Y_{11} (“Weil triviality” [5]) $I_{12} = dY_{11}$. As has been shown in [5] such a G -invariant Y_{11} would exist if one could impose on the Yang-Mills and gravitational supercurvatures the constraints $F_{\alpha\beta} = 0$, $R_{\alpha\beta a}{}^b = 0$. While the first constraint is always available it is known that for the coupled SUGRA–SYM theory $R_{\alpha\beta a}{}^b$ is intrinsically non vanishing, in the sense that it can not be set to zero by any field redefinition [8,9]. In the absence of such a constraint until now it was not possible to prove Weil triviality of I_{12} for an arbitrary gauge group H .

However, and this is the key observation of this letter, the theory at hand contains a two-form potential B among its physical fields with associated G -invariant curvature H defined as

$$\begin{aligned} H &= dB + X_{0,3} \\ dH &= X_4 = \frac{1}{30} \text{Tr} F^2 - \text{tr} R^2 \\ \delta_G B &= -X_{1,2} \\ \delta_G H &= 0. \end{aligned} \tag{24}$$

The second crucial point is that a consistent exact solution of (24) in *superspace* has been given in [10]. This means that all unphysical fields have been eliminated and that the remaining physical fields are subjected to SUSY transformation rules which represent correctly the algebra of local supersymmetry transformations, and assure in particular the nilpotency of the corresponding BRST operator. Using (24) it is now easy to prove Weil triviality for I_{12} , in fact:

$$I_{12} = X_4 X_8 = dH X_8 = d(H X_8) = \hat{d}(H X_8), \tag{25}$$

where the last step stems from the fact that $\delta_G(H X_8) = 0$. Now we can proceed along the lines of [5]. (23) and (25) imply

$$0 = \hat{d} \left(\frac{2}{3} X_4 Q_7 + \frac{1}{3} Q_3 X_8 - H X_8 \right) \equiv \hat{d} Q. \tag{26}$$

As above we decompose Q in sectors with definite ghost number p , $Q = \sum_{p=0}^{11} Q_{p,11-p}$, and (26) implies then the descent equations:

$$\begin{aligned} dQ_{0,11} &= 0 \\ dQ_{1,10} + \delta_G Q_{0,11} &= 0 \\ dQ_{2,9} + \delta_G Q_{1,10} &= 0. \end{aligned} \tag{27}$$

Applying i_ξ twice to the first equation in (27) and once to the second and recalling the definition of the Lie derivative \mathcal{L}_ξ and (11) we get:

$$\begin{aligned} \delta_S(i_\xi Q_{0,11}) - \frac{1}{2}d i_\xi i_\xi Q_{0,11} &= 0 \\ d(i_\xi Q_{1,10}) - \delta_S Q_{1,10} + \delta_G(i_\xi Q_{0,11}) &= 0 \\ dQ_{2,9} + \delta_G Q_{1,10} &= 0. \end{aligned} \tag{28}$$

Defining now the integral of a ten-superform ψ_{10} over ordinary space–time as

$$\int \psi_{10} \equiv \frac{1}{10!} \int d^{10}x \varepsilon^{m_1 \dots m_{10}} \psi_{m_1 \dots m_{10}}(z) \Big|_{\vartheta=0} \tag{29}$$

and noting that the integral of an exact ten-superform is zero, $\int d\psi_9 = 0$, we can integrate eqs. (28) and obtain a solution of the coupled cohomology problem (4) with:

$$\begin{aligned} \mathcal{A}_G &= \int Q_{1,10} \\ \mathcal{A}_S &= - \int i_\xi Q_{0,11}. \end{aligned} \tag{30}$$

It is easy to extract from (26), using (20), the components of Q with ghost number 1 and 0:

$$\mathcal{A}_G = \int \left(\frac{2}{3} X_4 X_{1,6} + \frac{1}{3} X_{1,2} X_8 \right) \tag{31}$$

$$\mathcal{A}_S = \int i_\xi \left(H X_8 - \frac{1}{3} X_{0,3} X_8 - \frac{2}{3} X_4 X_{0,7} \right). \tag{32}$$

Eq. (31) is clearly the usual expression of the ABBJ G -anomaly [3], while eq. (32) represents the supersymmetry anomaly we searched for. Using eqs. (21), (22) and (24), we can rewrite it as follows:

$$\begin{aligned} \mathcal{A}_S &= \int i_\xi \left(d(BX_8) + \frac{2}{3} (X_{0,3} X_8 - X_4 X_{0,7}) \right) \\ &= \Omega_S \int \left(-BX_8 - \frac{2}{3} X_{0,3} X_{0,7} \right) \\ &\equiv \Omega_S \Delta \end{aligned} \tag{33}$$

meaning that \mathcal{A}_S is a trivial cocycle of Ω_S . But it is easy to recognize Δ as the famous Green–Schwarz counterterm which cancels the G -anomaly \mathcal{A}_G (use again eqs. (21),(22))

$$\mathcal{A}_G = \Omega_G \Delta. \tag{34}$$

This means that if we redefine the effective action Γ according to $\tilde{\Gamma} = \Gamma - \Delta$ we obtain a G -invariant and supersymmetric theory. This is what one expects due to the fact that the anomaly \mathcal{A}_S gets induced by the gauge anomaly, and one can argue that the cancellation of the latter implies also the vanishing of the former. Clearly our argument does not exclude the presence of other (non trivial) supersymmetry cocycles which are not related to the ABBJ anomaly. This would, however, spoil the quantum consistency of the theory.

As a last remark we point out that the absence of SUSY anomalies in the $N = 1, D = 10$ SUGRA-SYM theory reflects the fact that also the related superstring theories are not plagued by SUSY anomalies.

Recently, in the literature a string-five-brane duality conjecture has been made [11,12]; it states that, in their critical space-time dimensions $D = 10$, superstrings (extended objects with one spatial dimension) are dual to super five-branes (extended objects with five spatial dimensions). In particular, in [13] the gauge and Lorentz anomalies of the heterotic five-brane sigma model in an $N = 1, D = 10$ SUGRA-SYM background have been determined by counting the chiral fermions and relying then on the index theorem. The net result is that the fourth-order polynomial responsible for the gauge and Lorentz anomalies of the sigma model is given precisely by X_8 (see (14)). To cancel these sigma model anomalies, instead of introducing a three-form H_3 satisfying $dH_3 = X_4$, one postulates the existence of a “dual” SUGRA-SYM theory based on a seven form H_7 [13] satisfying

$$dH_7 = X_8. \tag{35}$$

Clearly, as is known, eq. (35) assures the cancellation of the ABBJ anomalies also in the “dual” supergravity theory [14]. In this case again a supersymmetric partner $\tilde{\mathcal{A}}_S$ of these anomalies arises, and our procedure to compute it and to show its harmlessness could be repeated in a straightforward way simply by interchanging the roles of X_4 and X_8 once a consistent solution of (35) in *superspace* is obtained. But such a solution is still missing and seems difficult to exist. A small inspect into the Bianchi identity (35) in superspace reveals that, to obtain a consistent solution, one has at least to give up the rigid supersymmetry preserving constraint

$$T_{\alpha\beta}{}^a = 2\Gamma_{\alpha\beta}^a \tag{36}$$

by introducing a 1050 irreducible representation of $SO(10)$, $W_{c_1\dots c_5}{}^a$, according to

$$T_{\alpha\beta}{}^a = 2\Gamma_{\alpha\beta}^a + (\Gamma^{c_1\dots c_5})_{\alpha\beta}W_{c_1\dots c_5}{}^a. \quad (37)$$

In fact, as has been shown in ref. [9], eq. (36) leads necessarily to a *Lorentz and gauge-invariant* H_7 satisfying $dH_7 = 0$. Even with (37), it is by no means obvious that one can solve the Bianchi identity (35) consistently; on the other hand, to our knowledge no supergravity theory is known in which (36) is violated. This points in the direction of a conflict between the heterotic five-brane and supersymmetry.

Acknowledgements

The authors would like to thank Mario Tonin for useful discussions.

References

- [1] M. B. Green and J. H. Schwarz, *Phys. Lett.* **149B** (1984) 117
- [2] S. J. Adler, *Phys. Rev.* **177** (1969) 2426; J. S. Bell and R. Jackiw, *Nucl. Phys.* **A60** (1969) 47; L. Bonora and P. Pasti, *Phys. Lett.* **132B** (1983) 75
- [3] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge (1987)
- [4] L. Bonora, P. Pasti and M. Tonin, *Phys. Lett.* **156B** (1985) 341; *Nucl. Phys.* **B261** (1985) 249; *Phys. Lett.* **167B** (1986) 191
- [5] L. Bonora, P. Pasti and M. Tonin, *Nucl. Phys.* **B286** (1987) 150
- [6] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Princeton University Press, Princeton (1983)
- [7] L. Bonora and P. Cotta-Ramusino, *Phys. Lett.* **107B** (1981) 87; R. Stora, *Cargèse Lectures* (1983); B. Zumino, *Les Houches Lectures* (1983); B. Zumino, Y. S. Wu and Z. Zee, *Nucl. Phys.* **B239** (1984) 477
- [8] C. M. Hull, *Phys. Lett.* **167B** (1986) 51
- [9] A. Candiello and K. Lechner, *Nucl. Phys.* **B412** (1994) 479
- [10] L. Bonora, P. Pasti and M. Tonin, *Phys. Lett.* **188B** (1987) 335; L. Bonora, M. Bregola, K. Lechner, P. Pasti and M. Tonin, *Int. J. Mod. Phys.* **A5**

- (1990) 461; R. D'Auria and P. Fré, *Phys. Lett.* **200B** (1988) 63; R. D'Auria, P. Fré, M. Raciti and F. Riva *Int. J. Mod. Phys.* **A3** (1988) 953; L. Castellani, R. D'Auria and P. Fré, *Phys. Lett.* **196B** (1987) 349
- [11] M. J. Duff, *Class. Quantum Grav.* **5** (1988) 189
- [12] A. Strominger, *Nucl. Phys.* **B343** (1990) 167
- [13] J. A. Dixon, M. J. Duff and J. C. Plefka, *Phys. Rev. Lett.* **69** (1992) 3009
- [14] A. Salam and E. Sezgin, *Phys. Scripta* **32** (1985) 283; S. J. Gates and H. Nishino, *Phys. Lett.* **157B** (1985) 157

**THE SUPERSYMMETRIC VERSION OF THE
GREEN–SCHWARZ ANOMALY CANCELLATION MECHANISM***

Antonio Candiello and Kurt Lechner

Dipartimento di Fisica, Università di Padova

and

Istituto Nazionale di Fisica Nucleare, Sezione di Padova

Italy

Abstract

The $N = 1$, $D = 10$ Supergravity–Super–Yang–Mills (SUGRA-SYM) theory is plagued by ABBJ gauge and Lorentz anomalies which are cancelled via the Green-Schwarz anomaly cancellation mechanism. Due to the fact that the ABBJ anomalies are not invariant under supersymmetry (SUSY) transformations one concludes that the theory is plagued also by a SUSY anomaly. For the gauge groups $SO(32)$ and $E_8 \times E_8$ we compute this SUSY anomaly, by solving a coupled cohomology problem, and we show that it can be cancelled by subtracting from the action the known Green–Schwarz counterterm, the same which cancels also the ABBJ anomaly, the expected result. Finally, we argue that the corresponding mechanism does not apply in the dual SUGRA-SYM, related to the heterotic five-brane.

* Supported in part by M.P.I. This work is carried out in the framework of the European Community Programme “Gauge Theories, Applied Supersymmetry and Quantum Gravity” with a financial contribution under contract SC1-CT92-D789.