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k-ANOMALIES AND SPACE-TIME SUPERSYMMETRY IN THE GREEN-SCHWARZ HETEROTIC SUPERSTRING[1]

A. Candiello, K. Lechner and M. Tonin

Dipartimento di Fisica, Università di Padova
and
Istituto Nazionale di Fisica Nucleare, Sezione di Padova
Italy

Abstract

The computation of κ-anomalies in the Green-Schwarz heterotic superstring sigma-model and the corresponding Wess-Zumino consistency condition constitute a powerful alternative approach for the derivation of manifestly supersymmetric string effective actions. With respect to the beta-function approach this technique presents the advantage that a result which is obtained with the computation of beta-functions at n loops can be obtained through the calculation of κ-anomalies at $n-1$ loops. In this paper we derive by a direct one-loop perturbative computation the κ-anomaly associated to the Yang-Mills Chern-Simons threeform and, for the first time, the one associated to the Lorentz Chern-Simons threeform. In the calculation we shall use a convenient set of constraints for the pure $N=1$, $D=10$ supergravity theory which is algebraically identical to the standard set of constraints for the pure $N=1$, $D=10$ super Yang-Mills theory. Contrary to what is often stated in the literature we show that the Lorentz κ-anomaly gets contributions from the integration over both the fermionic *and* bosonic degrees of freedom of the string. A careful analysis of the absolute coefficients of all these anomalies reveals that they can be absorbed by setting $dH = \frac{\alpha'}{4}(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2)$, where α' is the string tension, the expected result. We show that this relation ensures also the absence of gauge and Lorentz anomalies in the sigma-model effective action. Moreover, the consistency condition of the κ-anomalies ensures the closure of the SUSY algebra in the Bianchi identities. We evidenciate the

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presence of infrared divergences in the heterotic string sigma model, which are due to the presence of the $d = 2$ scalar massless fields of the string, and present a conjecture for their cancellation which is intimately related to the locality and Wess-Zumino consistency of the κ-anomalies.

1 Introduction and summary

In its original formulation, the Neveu-Schwarz-Ramond (NSR) formulation, superstring theory appears manifestly Lorentz-covariant in its critical $D = 10$ dimension, while its principal drawback is the missing *manifest* target space-time supersymmetry. Its alternative formulation, the Green-Schwarz (GS) formulation, on the other hand exhibits manifest $D = 10$ space-time supersymmetry but, despite a lot of efforts, no manifestly Lorentz-covariant quantization scheme has been found until now. This difficulty is due to the fact that κ-invariance, the fundamental symmetry of the GS-string, cannot be fixed in a manifestly Lorentz-covariant way.

In the low energy limit superstring theory reduces to an $N = 1$, $D = 10$ Supergravity-Super-Yang-Mills (SUGRA-SYM) theory whose dynamics is described by appropriate effective actions. In the past such effective actions have been derived directly from the string amplitudes (see e.g. [1]) or by imposing the vanishing of the beta functions in string sigma models embedded in the zero modes of the string (see e.g. [11, 16]). These methods have been carried out almost exclusively in the NSR formulation, and as such they miss manifest space-time supersymmetry, but there is also some important work carried out in the GS framework [3, 4, 8].

Strings in the GS formulation on the other hand furnish an approach for the derivation of manifestly supersymmetric effective actions which relies neither on the knowledge of string amplitudes nor on the computation of beta-functions but on the fundamental κ-invariance of the GS-string. It goes as follows [2]. One writes a string sigma model action embedded in the ten-dimensional SUGRA-SYM superspace describing the massless modes of the underlying string theory. The κ-invariance of this action at the classical level implies constraints on the background supercurvatures and torsions which via the Bianchi identities lead to the equations of motion for the background fields; for the heterotic string, under investigation in this paper, at zeroth order in the string coupling constant α' these constraints describe the pure minimal SUGRA decoupled from SYM and the flat

$N = 1$, $D = 10$ SYM theories. If one quantizes the sigma model, κ-anomalies can show up, whose form is strongly restricted by the Wess-Zumino (WZ) consistency condition. It turns out [2] that the non trivial solutions of the corresponding cohomology problem are all such that the related κ-anomalies can be absorbed by suitably modifying the (classical) constraints on the supercurvatures and torsions of the background fields. This procedure has to be carried out order by order in α', i.e. loop by loop in the quantum expansion of the sigma model. The solution of the Bianchi identities with these new constraints gives the new equations of motion and hence the string-corrected effective action for SUGRA-SYM as a power series in α' with manifest SUSY.

In this paper we want to illustrate the powerfulness and conceptual elegance of this procedure in the heterotic string by performing in particular for the first time the direct perturbative computation of the κ-anomaly related to the Lorentz-Chern-Simons term, but we shall also evidenciate its technical problems and conceptual limitations among which the most striking one is the appearance of infrared divergences.

With respect to the beta-function approach our algorithm presents a decisive advantage: a contribution to the effective action obtained with beta-functions at the n-th loop order is obtained with our algorithm at $(n-1)$ loops. May be this is the reason why the Lorentz-Chern-Simons term has never been derived using beta-functions (a two-loop computation!) while the κ-anomaly implying this term arises at one-loop. Another technical difficulty related with the beta-function approach is that the absence of a manifestly Lorentz covariant quantization procedure gives rise to Lorentz non-covariant intermediate results which are not easy to handle. The κ-anomaly algorithm, on the other hand, produces directly the modified constraints on the superspace so that the equations of motion can be derived in a straightforward way by solving the Bianchi identities via standard techniques, and the Lorentz non-covariance of the quantization procedure can be easily handled, at least in the computation of the Lorentz κ-anomaly.

The paper is organized as follows. In section II we present the pure Supergravity and Super Yang-Mills system which constitutes the background of the

heterotic Green-Schwarz string, along with the constraints on the supercurvatures and torsions needed to set the decoupled theory on-shell. The set of these constraints is not unique, but is determined modulo field redefinitions. Using this freedom we choose for the SUGRA a set of constraints in which the purely spinorial components of the Lorentz-curvature vanish, $R_{\alpha\beta a}{}^{b} = 0$ [18, 20]. This is possible at the classical level where SUGRA is decoupled from SYM. This constraint being analogous to the constraint for the SYM curvature, $F_{\alpha\beta} = 0$ [30], when computing the Lorentz κ-anomaly we can, to a certain extent but with an important difference, follow the procedure for the computation of the Yang-Mills κ-anomaly.

In section III we present the action for the Green-Schwarz heterotic sigma model embedded in the SUGRA-SYM background together with its symmetries. The action is κ-invariant only if the constraints that pose the background fields on-shell are satisfied. The action is also invariant under gauge and Lorentz combined transformations of the background fields and the string fields. These transformations, as we will see, give rise to "anomalies", but we would like to point out that these transformations, not being actually symmetries of the theory, do not produce true anomalies: they are a useful tool for the analysis of the related gauge- and Lorentz-type κ-anomalies which are true anomalies of the theory.

In section IV we discuss briefly the quantization and describe the normal coordinate expansion of the Green-Schwarz action in the framework of the Batalin-Vilkovisky approach.

In section V we rederive the gauge anomaly and the κ-anomaly associated to the Yang-Mills Chern-Simons form. Usually when computing an anomaly one regularizes the classical action, computes the associated effective action and gets the anomaly by varying the effective action. A less known alternative method consists in regularizing the classical action and computing the variation of the regularized classical action to get an "anomalous vertex". The anomaly is simply obtained by inserting the anomalous vertex in all Feynman diagrams and by keeping only those which survive when the regulator goes to zero. We shall use this alternative method to compute the gauge anomaly and the κ-anomaly

associated to the Yang-Mills Chern-Simons form.

In section VI we apply a κ-gauge fixing to the expanded action which breaks the manifest Lorentz invariance of the theory, but leaves a residual $SO(8)$ invariance there. In analogy to the computation of section V we identify the anomalous vertex associated to Lorentz transformations and compute the corresponding Lorentz anomaly together with its absolute coefficient. It turns out that only the fermionic degrees of freedom of the string contribute to the Lorentz anomaly, and that its coefficient, with respect to the naive guess, is divided by a factor of two. This is due to the fact that the κ-symmetry implies that half of the 16 fermionic degrees of freedom of the string are unphysical and therefore only 8 of them circulate in the anomalous diagram. The final result can be easily Lorentz-covariantized by employing the manifest $SO(8)$ invariance of the result. The gauge and Lorentz anomalies computed in sections V and VI can be eliminated by associating to the two-superform potential B of the $N = 1$, $D = 10$ supergravity sector transformation properties such that its curvature, defined as $H = dB + \frac{\alpha'}{4}(\omega_{3YM} - \omega_{3L})$ where ω_{3YM} and ω_{3L} are the Yang-Mills and Lorentz Chern-Simons forms, is gauge and Lorentz invariant, as one expects.

Section VII is devoted to the computation of the Lorentz κ-anomaly. In the Yang-Mills sector the κ-transformation acts essentially as a field-dependent gauge transformation and therefore the computation of the κ-anomaly is closely related to that of the gauge anomaly. The action of the κ-transformation in the gravitational sector is, however, a combination of a field-dependent local Lorentz transformation and an "intrinsic" κ-transformation and the relation between the Lorentz κ-anomaly $\mathcal{A}_\kappa$ and the Lorentz anomaly $\mathcal{A}_L$ is less obvious. In fact $\mathcal{A}_L$ gets contributions only from loops where the fermionic fields of the string circulate, while the κ-anomaly $\mathcal{A}_\kappa$ gets contributions also from loops with the bosonic fields of the string circulating. These loops are necessary to saturate the

coupled cohomology problem

$$\begin{aligned} \Omega_L \mathcal{A}_L &= 0 \\ \Omega_\kappa \mathcal{A}_L + \Omega_L \mathcal{A}_\kappa &= 0 \\ \Omega_\kappa \mathcal{A}_\kappa &= 0. \end{aligned} \tag{1}$$

where Ω_κ and Ω_L are the BRS operators associated to the κ-transformations and Lorentz transformations respectively. Here the situation is similar to that found in the case of the SUSY anomaly $\mathcal{A}_S$ in a supersymmetric chiral Yang-Mills theory. In that case the presence of an ABBJ Yang-Mills anomaly $\mathcal{A}_G$ induces the presence of a SUSY anomaly via the coupled cohomology problem:

$$\begin{aligned} \Omega_G \mathcal{A}_G &= 0 \\ \Omega_S \mathcal{A}_G + \Omega_G \mathcal{A}_S &= 0 \\ \Omega_S \mathcal{A}_S &= 0, \end{aligned} \tag{2}$$

where Ω_G and Ω_S are the BRS operators associated to gauge and SUSY transformations respectively. As is well known the ABBJ anomaly $\mathcal{A}_G$ gets contributions only from one-loop diagrams where chiral quarks circulate, while the SUSY anomaly $\mathcal{A}_S$ gets contributions also from loops of squarks, the scalar bosonic superpartners of the quarks. These loops with squarks are necessary to saturate the coupled cohomology problem (2).

We derive $\mathcal{A}_\kappa$ through the standard procedure by identifying the relevant part of the effective action, by integrating over fermions *and* bosons and by varying it. Besides the local terms which saturate exactly (1) one gets infrared divergences coming from the integration over the *massless* bosons which are non local and which spoil, moreover, κ-invariance in the sense that they would give rise to non-local κ-anomalies. This is clearly related to the fact that there exists no κ-symmetry preserving infrared regularization procedure for the GS sigma model: as it stands, the perturbative expansion of the GS sigma model effective action is inconsistent due to the presence of these infrared divergences. We argue by exhibiting an explicit simplified example that these divergences are actually due to an intrinsic non analyticity of the effective action, as a functional of the fields,

which can therefore not be expanded perturbatively as a polynomial in the external fields. Assuming that a non-perturbative treatment will eventually eliminate these divergences we can invoke a) the Wess-Zumino consistency condition for the κ-anomalies and b) their locality to eliminate them completely without arbitrariness left. But this recipe amounts to a conjecture and not to a solution of the infrared problem.

The κ-anomalies derived in this way induce a background SUGRA-SYM theory based on the Bianchi identity in superspace

$$dH = \frac{\alpha'}{4}\left(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2\right) \tag{3}$$

precisely as predicted by the Green-Schwarz anomaly cancellation mechanism in $N = 1$, $D = 10$ SUGRA-SYM [23] and by the (non supersymmetric) effective action derived directly by the Veneziano-like superstring amplitudes [1].

The κ-anomaly method produces automatically the constraints on the background fields with which one has to solve (3) once the WZ consistency condition is satisfied. In section VIII we check that our κ-anomalies satisfy indeed the WZ condition and determine the corresponding superspace constraints. Differences between these constraints and other constraints in the literature [15] are shown to be related to κ-cocycles trivial at one loop.

Section IX contains some conclusions and outlooks on the κ-anomaly computation at higher loop orders, together with a brief analysis of the open problems in the quantization procedure and perturbative treatment of the GS string sigma model.

2 Pure Supergravity and Super Yang-Mills

In this section we outline the background theory required by the Green-Schwarz heterotic sigma model.

A superspace in ten dimensions is parametrized by the coordinates $Z^M(\sigma) = (X^m(\sigma), \vartheta^\mu(\sigma))$, where X^m $(m = 0, 1, \ldots, 9)$ are the bosonic degrees of freedom

and ϑ^μ ($\mu = 1, \ldots, 16$) are the fermionic degrees of freedom. The supervielbein one-form $E^A = dZ^M E_M{}^A(Z)$ describes the local flat frame ($A = (a, \alpha)$, where ($a = 0, 1, \ldots, 9$; $\alpha = 1, \ldots, 16$), is a flat index). For the ten-dimensional local Lorentz group we use a Minkowski metric η_{ab} with signature -8. The $SO(32)$ Lie-valued Yang-Mills connection one-superform is $A = E^B A_B(Z)$, while the Lorentz-valued connection one-superform is $\Omega_A{}^B = E^C \Omega_{CA}{}^B(Z)$, where $\Omega_a{}^\alpha = \Omega_\alpha{}^a = 0$, $\Omega_\alpha{}^\beta = \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta \Omega_{ab}$. The supergravity potentials also comprehend the two-superform $B = \frac{1}{2} E^C E^D B_{DC}(Z)$. The field strengths associated to E^A, B, A and $\Omega_A{}^B$ are given by

$$T^A = DE^A = dE^A + E^B \Omega_B{}^A \tag{4a}$$

$$W = dB \tag{4b}$$

$$F = dA + AA \tag{4c}$$

$$R_A{}^B = d\Omega_A{}^B + \Omega_A{}^C \Omega_C{}^B \tag{4d}$$

and the corresponding Bianchi identities are

$$DT^A = E^B R_B{}^A \tag{5a}$$

$$DW = 0 \tag{5b}$$

$$\mathcal{D}F = 0 \tag{5c}$$

$$DR_A{}^B = 0, \tag{5d}$$

where $d = dZ^M \partial_M$, D is the Lorentz covariant superdifferential and $\mathcal{D}$ is the gauge covariant superdifferential. The pure supergravity and Yang-Mills theories are set on-shell by imposing a minimal set of constraints on the curvatures, which is uniquely determined modulo field redefinitions, and we choose it to be

$$T_{\alpha\beta}{}^a = 2\Gamma^a_{\alpha\beta}, \qquad T_{\alpha a}{}^b = 0 \tag{6a}$$

$$(dB)_{a\alpha\beta} = 2(\Gamma_a)_{\alpha\beta}, \qquad (dB)_{\alpha\beta\gamma} = 0, \qquad (dB)_{\alpha ab} = 0 \tag{6b}$$

$$F_{\alpha\beta} = 0 \tag{6c}$$

$$R_{\alpha\beta a}{}^b = 0. \tag{6d}$$

Note in particular the constraint (6d): as shown in [20, 2], it can always be imposed for pure supergravity. This constraint allows to maintain a close parallelism between the gauge and gravitational sectors.

The Bianchi identities then imply [18]

$$T_{\alpha\beta}{}^{\gamma} = 2\delta^{\gamma}_{(\alpha}\lambda_{\beta)} - (\Gamma_g)_{\alpha\beta}(\Gamma^g)^{\gamma\varepsilon}\lambda_{\varepsilon} \tag{7a}$$

$$T_{a\alpha}{}^{\beta} = \frac{1}{4}(\Gamma^{bc})_{\alpha}{}^{\beta}T_{abc} \tag{7b}$$

$$W_{abc} \equiv (dB)_{abc} = T_{abc} \tag{7c}$$

$$D_{\alpha}\lambda_{\beta} = -(\Gamma^g)_{\alpha\beta}D_g\phi + \lambda_{\alpha}\lambda_{\beta} + \frac{1}{12}(\Gamma^{abc})_{\alpha\beta}T_{abc} \tag{7d}$$

$$D_{\alpha}T_{abc} = -6(\Gamma_{[a})_{\alpha\varepsilon}T_{bc]}{}^{\varepsilon} \tag{7e}$$

$$F_{a\alpha} = 2(\Gamma_a)_{\alpha\varepsilon}\chi^{\varepsilon} \tag{7f}$$

$$R_{a\alpha bc} = 2(\Gamma_a)_{\alpha\varepsilon}T_{bc}{}^{\varepsilon} \tag{7g}$$

$$D_{\alpha}\chi^{\beta} = \frac{1}{4}(\Gamma^{ab})_{\alpha}{}^{\beta}F_{ab} + T_{\alpha\varepsilon}{}^{\beta}\chi^{\varepsilon} \tag{7h}$$

$$D_{\alpha}T_{cd}{}^{\beta} = \frac{1}{4}(\Gamma^{ab})_{\alpha}{}^{\beta}R_{abcd} + T_{\alpha\varepsilon}{}^{\beta}T_{cd}{}^{\varepsilon}. \tag{7i}$$

Here χ^{ε} and $T_{ab}{}^{\varepsilon}$ are the gluino and the gravitino field strengths, T_{abc}, the vectorial part of the torsion, is completely antisymmetric in its indices, ϕ is the dilaton superfield and the gravitello superfield is $\lambda_{\alpha} \equiv D_{\alpha}\phi$. Note the symmetry between the gauge and Lorentz sector visible in the last four equations. The computation of the related equations of motion can now be performed (see for example, with constraints slightly different from ours, Ref. [15]), but for the purposes of this work we do not need them.

It is also useful to introduce the gauge and Lorentz Chern-Simons three-superforms

$$\begin{aligned} \omega_{3YM} &= \mathrm{tr}\left(AF - \frac{1}{3}A^3\right) \\ \omega_{3L} &= \mathrm{tr}\left(\Omega R - \frac{1}{3}\Omega^3\right) \end{aligned} \tag{8}$$

satisfying

$$\begin{aligned} d\omega_{3YM} &= \mathrm{tr}(FF) \\ d\omega_{3L} &= \mathrm{tr}(RR) \end{aligned} \tag{9}$$

which will play a central role in what follows. In (8), (9) the traces are in the fundamental representations of $SO(32)$ and $SO(10)$ respectively.

3 The action and its symmetries

The action for the heterotic Green-Schwarz sigma model in a SUGRA-SYM background is given by [26, 25]

$$I = -\frac{1}{2}\int d^2\sigma \left(\sqrt{g}g^{ij}V_i{}^aV_{ja} + \varepsilon^{ij}V_i{}^CV_j{}^DB_{DC} - \sqrt{g}e_-{}^j\psi\mathcal{D}_j\psi\right). \tag{10}$$

Our notations are as follows. The string worldsheet is parametrized by the coordinates σ^i $(i,j=0,1)$. The sigma-model fields are the zweibeins $e_\pm{}^i(\sigma)$ with $e_i{}^\pm$ its inverses, the superspace coordinates $Z^M(\sigma)$ which are worldsheet scalars and the 32 Majorana-Weyl heterotic world-sheet fermions $\psi^r(\sigma)$ $(r=1,\ldots,32)$ which stay in the fundamental representation of $SO(32)$. $\mathcal{D}_j\psi = (\partial_j - A_j)\psi$, where $A_j = V_j{}^BA_B$ and the induced supervielbein $V_i{}^A$ is defined as $V_i{}^A = \partial_iZ^ME_M{}^A$. In the following we shall use flat light-cone indices on the worldsheet defined by $W_\pm = e_\pm{}^iW_i$ if W_i is a worldsheet vector. The worldsheet metric is $g_{ij}(\sigma)$, with g^{ij} its inverse and $g = -\det g_{ij}$ and ε^{ij} is the antisymmetric Ricci tensor. The metric and the Ricci tensor can be expressed in terms of the zweibeins through:

$$\begin{aligned} g^{ij} &= \frac{1}{2}\left(e_-{}^ie_+{}^j + e_+{}^ie_-{}^j\right) \\ \frac{\varepsilon^{ij}}{\sqrt{g}} &= \frac{1}{2}\left(e_-{}^ie_+{}^j - e_+{}^ie_-{}^j\right). \end{aligned} \tag{11}$$

The self-dual projector $P^{ij} = g^{ij} + \varepsilon^{ij}/\sqrt{g}$ can be expressed through the zweibeins as $P^{ij} = e_-{}^ie_+{}^j$. By introducing the two-dimensional Dirac matrices γ^p in a Majorana representation $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ such that $\gamma^3 = -\gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and using two-component Majorana spinors ψ to describe the heterotic fermions, the last term in (10) can be written as

$$I_H = \frac{1}{2}\int d^2\sigma\sqrt{g}e_-{}^j\overline{\psi}\gamma_+\mathcal{D}_j\psi = \frac{1}{2}\int d^2\sigma\sqrt{g}e_p{}^j\overline{\psi}\gamma^p\frac{1+\gamma_3}{2}\mathcal{D}_j\psi, \tag{12}$$

where $\gamma_\pm = \gamma^0 \pm \gamma^1$ and $\overline{\psi} = \psi^T\gamma^0$. For notational simplicity we use the same symbol for one-component spinors since no confusion should arise. The second term in (12) will be used in the following.

The action (10) is invariant under $d = 2$ diffeomorphisms and local $d = 2$ Lorentz and Weyl transformations. In addition the Green-Schwarz action is also invariant under Siegel's local κ-symmetry [32] which permits to eliminate half of the 16 ϑ^μ. The transformation parameter is a (self-dual) world-sheet vector and space-time spinor $\kappa_{+\beta}(\sigma)$. The string fields transform as follows:

$$\delta_\kappa Z^M = \Delta^\alpha E_\alpha{}^M \tag{13a}$$

$$\delta_\kappa \psi = \Delta^\alpha A_\alpha \psi \equiv C\psi \tag{13b}$$

$$\delta_\kappa e_+{}^i = -4e_-{}^i \left(V_+{}^\varepsilon - \frac{1}{2}\psi\chi^\varepsilon\psi \right) \kappa_{+\varepsilon} \tag{13c}$$

$$\delta_\kappa g = \delta_\kappa e_-{}^i = 0 \tag{13d}$$

where

$$\Delta^\alpha = V_-{}^a(\Gamma_a)^{\alpha\beta}\kappa_{+\beta} \equiv (\not{V}_-\kappa_+)^\alpha; \tag{14}$$

we use the notation $\not{W} \equiv W_a(\Gamma^a)_{\alpha\beta}$ for a vector field W_a. Correspondingly it can be seen that the target superfields and superforms transform as

$$\delta_\kappa V_i{}^A = D_i\Delta^\alpha\delta_\alpha{}^A + V_i{}^B\Delta^\gamma T_{\gamma B}{}^A - V_i{}^B L_B{}^A \tag{15a}$$

$$\delta_\kappa B_{MN} = \delta_\kappa Z^L \partial_L B_{MN} \tag{15b}$$

$$\delta_\kappa T_{A\cdots}{}^{B\cdots} = \Delta^\alpha \partial_\alpha T_{A\cdots}{}^{B\cdots} \tag{15c}$$

$$\delta_\kappa A_i = \mathcal{D}_i C + F_i \tag{15d}$$

$$\delta_\kappa \Omega_{ia}{}^b = D_i L_a{}^b + R_{ia}{}^b \tag{15e}$$

where we defined:

$$\Omega_{ia}{}^b = \partial_i Z^M \Omega_{Ma}{}^b \tag{16a}$$

$$L_a{}^b = \Delta^\gamma \Omega_{\gamma a}{}^b \tag{16b}$$

$$F_i = V_i{}^B \Delta^\alpha F_{\alpha B} \tag{16c}$$

$$R_{ia}{}^b = V_i{}^B \Delta^\alpha R_{\alpha Ba}{}^b. \tag{16d}$$

Under κ-transformations the action varies as:

$$\delta_\kappa I = -\frac{1}{2}\int d^2\sigma\Big(2\sqrt{g}g^{ij}V_{ia}V_j{}^B\Delta^\gamma T_{\gamma B}{}^a + \varepsilon^{ij}V_i{}^C V_j{}^D\Delta^\gamma(dB)_{\gamma DC}$$
$$+\sqrt{g}e_-{}^j\psi F_j\psi - 4\sqrt{g}V_-^2\left(V_+{}^\varepsilon - \frac{1}{2}\psi\chi^\varepsilon\psi\right)\kappa_{+\varepsilon}\Big). \tag{17}$$

The vanishing of the purely gravitational contribution in (17) requires precisely the constraints (6a), (6b). With these constraints the gravitational part of (17) becomes, in fact,

$$(\delta_\kappa I)_{\rm grav} = 2\int d^2\sigma\left(\sqrt{g}V_-^2V_+{}^\alpha\kappa_{+\alpha} - \sqrt{g}V_+{}^\alpha(\not{V}_-\Delta)_\alpha\right) \tag{18}$$

which vanishes since from the definition (14) of Δ^α, one has

$$(\not{V}_-\Delta)_\alpha = V_-^2\kappa_{+\alpha}, \qquad V_-^2 \equiv V_-{}^aV_{-a}. \tag{19}$$

The vanishing of the Yang-Mills contribution in (17) requires the vanishing of the spinor-spinor component of the Yang-Mills curvature (6c). Indeed with the aid of (6c) and (7f) we get

$$F_i = -2V_i^b\Delta^\alpha(\Gamma_b)_{\alpha\beta}\chi^\beta \tag{20}$$

and, due to (19),

$$F_- = -2V_-^2(\kappa_{+\varepsilon}\chi^\varepsilon). \tag{21}$$

Notice that κ-invariance, at the classical level, does not imply any particular constraint on the spinor-spinor components of the Lorentz curvature two-superform $R_{\alpha\beta}$. There are, in fact, a lot of field redefinitions which keep the constraints in (6a) and (6b) invariant and give rise to different choices for $R_{\alpha\beta a}{}^b$. The constraint (6d) is extremely convenient for the purpose of the computation of the Lorentz κ-anomaly. It allows to follow as closely as possible the derivation of the Yang-Mills κ-anomaly. With this respect we notice that the relations (6d) and (7g) imply in complete analogy to the Yang-Mills case that

$$R_{iab} = -2V_i^c\Delta^\alpha(\Gamma_c)_{\alpha\beta}T_{ab}{}^\beta \tag{22}$$

and therefore, due to (19)

$$R_{-ab} = -2V_-^2\kappa_{+\varepsilon}T_{ab}{}^\varepsilon \tag{23}$$

which is proportional to V_-^2, exactly as in (21). Eq. (23) will be of fundamental importance in the derivation of the Lorentz κ-anomaly.

4 Quantization and normal coordinate expansion

A preliminary step to quantize the sigma-model action considered in the previous section is to gauge-fix its local symmetries. Since the algebra is open and reducible (in fact infinitely reducible) the safest way to do that is to work in the Batalin-Vilkovisky (BV) approach [17]. Calling ϕ^I all the fields, ghosts, antighosts, Lautrup-Nakanishi fields and secondary ghosts of the model, one introduces for each ϕ^I an antifield ϕ^*_I with statistics opposite to ϕ^I and writes the extended action $S_0[\phi, \phi^*]$

$$S_0[\phi, \phi^*] = I[\phi] + (-1)^{n(I)} \phi^*_I \Delta^I[\phi, \phi^*]. \tag{24}$$

Here $n(I)$ is the grading of ϕ^I. $I[\phi] = S_0[\phi, 0]$ is just the action in (10), and the terms linear in ϕ^*_I are obtained by coupling the antifields to the BRS transformations of the fields and the higher order terms are chosen so that S_0 satisfies the master equation

$$(S_0, S_0) \equiv (-1)^{n(I)} \frac{\delta S_0}{\delta \phi^I} \frac{\delta S_0}{\delta \phi^*_I} = 0. \tag{25}$$

As usual, here and in the following, repeated indices I imply sums over discrete indices and integration over worldsheet coordinates. Notice that the BRS transformations of ϕ are

$$\delta \phi^I = (S_0, \phi^I)\Big|_{\phi^*=0} = (-1)^{n(I)} \frac{\delta S_0}{\delta \phi^*_I}\Big|_{\phi^*=0} = \Delta^I[\phi, 0]. \tag{26}$$

The formalism is a graded canonical one with ϕ, ϕ^* as conjugate variables and

$$(\mathcal{F}, \mathcal{G}) = (-1)^{n(I)} \left(\frac{\delta \mathcal{F}}{\delta \phi^I} \frac{\delta \mathcal{G}}{\delta \phi^*_I} + \frac{\delta \mathcal{F}}{\delta \phi^*_I} \frac{\delta \mathcal{G}}{\delta \phi^I} \right) \tag{27}$$

as graded Poisson bracket, $\mathcal{F}$ and $\mathcal{G}$ being even functionals of ϕ, ϕ^* with zero ghost number. The gauge-fixing is realized through a canonical transformation on

$S_0[\phi, \phi^*]$, generated by a suitably chosen "gauge fermion", $\Psi[\phi]$ of ghost number -1. We do not report here the explicit form of the extended action $S_0[\phi, \phi^*]$ for our heterotic string sigma-model. It can be found for instance in the last paper of Ref. [2], Eq. (3.6).

On the other hand, calculations of quantum effective actions are simplified by using the background field technique. It consists in performing, before doing the gauge fixing, a split of the field variables ϕ^I into a classical part ϕ_0^I and their "fluctuations" χ^I to be quantized. In order to maintain local Lorentz and gauge invariance we shall adopt a variant of this method known as "normal coordinate expansion" [5, 6]. In that case the splitting is

$$\phi^I = \phi_0^I + \Phi^I(\phi_0, \chi) \tag{28}$$

where $\chi^{\hat{I}}$ are the quantum fields. More precisely let us divide the set of fields ϕ^I in four groups

$$\phi^I \equiv \left(q^i(\sigma), \psi^r(\sigma), Z^M(\sigma), k_n^{\hat{\alpha}}(\sigma)\right) \tag{29}$$

where Z^M are the string supercoordinates, ψ^r the heterotic fermions, q^i denote fields, ghosts etc that are inert under Lorentz and gauge transformations and $k_n^{\hat{\alpha}}$ are ghosts and LN fields that transform as (left-handed or right-handed) Lorentz spinors (i.e. $\hat{\alpha}$ denotes an upper or lower index α).

Similarly

$$\phi_0^I \equiv \left(q_0^i(\sigma), \psi_0^r(\sigma), Z_0^M(\sigma), k_{0n}^{\hat{\alpha}}(\sigma)\right) \tag{30}$$

and

$$\chi^{\hat{I}} \equiv \left(Q^i(\sigma), \Psi^r(\sigma), y^A(\sigma), \kappa_n^{\hat{\alpha}}(\sigma)\right). \tag{31}$$

Then Eq. (28) writes

$$q^i = q_0^i + Q^i \tag{32a}$$

$$Z^M = Z_0^M + \Pi^M(Z_0, y) \tag{32b}$$

$$\psi = e^{\Lambda(Z_0,y)}\left(\psi_0 + \Psi\right) \tag{32c}$$

$$k_n = e^{\Sigma(Z_0,y)}\left(k_{0n} + \kappa_n\right) \tag{32d}$$

where Π^M, Λ and Σ depend on Z_0^M and y^A only, Λ being $SO(32)$ Lie algebra valued and Σ Lorentz valued. In particular for the zweibein we write (32a) as

$$e_{\pm}{}^{i} = e_{0\pm}{}^{i} + h_{\pm}{}^{i}. \tag{33}$$

It is possible to implement the normal coordinate expansion in the framework of the BV approach, as will be seen elsewhere [13]. For our purposes it is sufficient to sketch the procedure.

First notice that, after the splitting (28), the action acquires an invariance under a local shift of the background fields ϕ_0, supplemented by a suitable transformation of the quantum fields χ. Then consider the action

$$\tilde{S}_0 = S_0[\phi, \phi^*] + (-1)^{n(I)} \phi^*_{0I} \mathcal{E}^I \tag{34}$$

where ϕ^*_{0I} are the antifields for ϕ_0^I and $\mathcal{E}^I$ are the (classical) local shift ghosts.

The next step is to perform on $\tilde{S}_0$ a canonical transformation of the fields ϕ, ϕ_0 and their (conjugate) antifields ϕ^*, ϕ_0^* to implement the transformation (28) on the fields ϕ^I, leaving unchanged the background fields ϕ_0^I. Then the gauge fixing is performed by means of a further canonical transformation generated by a suitable gauge fermion to obtain the final extended classical action $S[\chi, \chi^*; \phi_0, \phi_0^*]$ where $\chi^*_{\hat{I}}$ are the antifields associated to $\chi^{\hat{I}}$.

Path-integrating over $\chi^{\hat{I}}$, one can define, by the standard procedure, the effective action $\tilde{\Gamma}[\chi, \chi^*; \phi_0, \phi_0^*]$ (as usual, the classical fields associated to the quantum fields $\chi^{\hat{I}}$ are still denoted $\chi^{\hat{I}}$).

Thanks to the shift symmetry, it is possible to perform on $\tilde{\Gamma}$ a canonical transformation to get an action $\hat{\Gamma}[\chi, \chi^*; \phi_0, \phi^*]$ where the terms linear in $\chi^{\hat{I}}$ are absent. Then by taking $\hat{\Gamma}$ at $\chi = 0 = \chi^*$ and $\mathcal{E} = 0$ one arrives at an effective action $\Gamma[\phi_0, \phi_0^*]$ that satisfies the Slavnov-Taylor identity

$$(\mathit{\Gamma}, \mathit{\Gamma}) = 0. \tag{35}$$

The field equations are

$$\frac{\delta \mathit{\Gamma}}{\delta \phi_0^I}[\phi_0, \phi_0^*] = 0. \tag{36}$$

At zeroth order in α', for $\phi_0^* = 0$ and disregarding the ghost fields one has the classical field equations

$$D_- V_{+a} + \overline{\psi}_0 \gamma_+ \left(V_-{}^\alpha (\Gamma_a)_{\alpha\beta} \chi^\beta + \frac{1}{2} V_-{}^c F_{ac} \right) \psi_0 = 0 \tag{37a}$$

$$\not{V}_{-\alpha\beta} \left(V_+{}^\beta - \frac{1}{2} \psi_0 \chi^\beta \psi_0 \right) = 0 \tag{37b}$$

$$\gamma_+ \left(\partial_- - A_- + \frac{1}{2\sqrt{g}} \partial_i (\sqrt{g} e_-{}^i) \right) \psi_0 = 0 \tag{37c}$$

$$V_-^2 = 0 \tag{37d}$$

$$V_+{}^a V_{+a} - \psi_0 \mathcal{D}_+ \psi_0 = 0. \tag{37e}$$

We will limit ourselves to perform one-loop computations for an on-shell configuration of the background fields ϕ_0 satisfying (37). Notice that in particular, due to the Virasoro constraint (37d), the vectors F_i, $R_{ia}{}^b$, appearing in the transformation of the connections, become chiral

$$\begin{aligned} g_0^{ij} F_j &= \frac{\varepsilon^{ij}}{\sqrt{g_0}} F_j, \qquad \text{i.e.} \qquad F_- = 0 \\ g_0^{ij} R_{ja}{}^b &= \frac{\varepsilon^{ij}}{\sqrt{g_0}} R_{ja}{}^b, \qquad \text{i.e.} \qquad R_{-a}{}^b = 0. \end{aligned} \tag{38}$$

The normal coordinate expansion amounts to a suitable choice of the functions $\Pi^M(Z_0, y)$, $\Lambda(Z_0, y)$ and $\Sigma(Z_0, y)$ in Eqs. (32) in such a way as to restore the Lorentz and gauge covariance of the expansion along the quantum fields of a functional like the action I, Eq. (10). The geometrical meaning of Π^M is that it defines the variables y^A so that y^A are tangent vectors to the geodesic joining the origin of the normal coordinate Z_0 to the point Z. For more details about Π^M and Λ see Ref. [5] and [7] respectively. Up to second order in y^A

$$\Pi^M = y^B E_B{}^M + \frac{1}{2} y^B y^C D_C E_B{}^M + o(y^3) \tag{39a}$$

$$\Lambda = y^B A_B + \frac{1}{2} y^B y^C D_C A_B + o(y^3), \tag{39b}$$

where D_C is the Lorentz covariant derivative. A scalar functional which, as the action (10), depends on Z^M only through $V_i{}^A(Z)$ and the flat components of

the connections and curvatures can now be expanded, according to the Mukhi algorithm,

$$I(Z,\psi,q)=\sum_{n=0}^{\infty}\frac{1}{n!}\Delta^n I(Z_0,\psi_0+\Psi,q_0+Q) \tag{40}$$

where the repeated application of the operator Δ is defined as follows

$$\Delta V_i{}^A = D_i y^A + V_i{}^B y^C T_{CB}{}^A \tag{41a}$$

$$\Delta \Omega_{iA}{}^B = V_i{}^C y^D R_{DCA}{}^B \tag{41b}$$

$$\Delta A_i = V_i{}^C y^D F_{DC} \tag{41c}$$

$$\Delta T_{A\cdots}{}^{B\cdots} = y^C \mathcal{D}_C T_{A\cdots}{}^{B\cdots} \tag{41d}$$

$$\Delta y^A = 0 \tag{41e}$$

$$\Delta(\psi_0+\Psi) = 0 \tag{41f}$$

$$\Delta(q_0+Q) = 0. \tag{41g}$$

Here $D_i y^A = \partial_i y^A + y^B \Omega_{iB}{}^A$ and $T_{A\cdots}{}^{B\cdots}$ is any Lorentz and Yang-Mills tensor. The expansion of $V_i{}^A(Z)$ up to second order in y^A is

$$\begin{aligned} V_i{}^A(Z) &= \partial_i Z^M E_M{}^A(Z) = \partial_i Z_0^M E_M{}^A + D_i y^A + \partial_i Z_0^M E_M{}^C y^B T_{BC}{}^A \\ &+ \frac{1}{2} D_i y^C y^B T_{BC}{}^A + \frac{1}{2}\partial_i Z_0^M E_M{}^D y^E T_{ED}{}^C y^B T_{BC}{}^A \\ &+ \frac{1}{2}\partial_i Z_0^M E_M{}^C y^B y^D D_D T_{BC}{}^A + \frac{1}{2} y^D \partial_i Z_0^M E_M{}^C y^B R_{BCD}{}^A + o(y^3). \end{aligned} \tag{42}$$

The fields on the last member of this expression are all evaluated in Z_0. The action is still BRS invariant after normal coordinate expansion if we maintain for the background fields ϕ_0 the classical variations (13), (15) and impose suitable transformation properties on the quantum fields. These latter can be read from the terms linear in Q^*, Ψ^*, y^*_A, in the action $S[\chi,\chi^*;\phi_0,\phi_0^*]$. However a convenient way to get $\delta_\kappa y^A$, $\delta_\kappa \Psi$, $\delta_\kappa h_p{}^i$ is the following. Consider the expansion of $\delta Z^M E_M{}^A(Z)$, obtained in analogy with the expansion of $V_i{}^A(Z)$ in Eq. (42) to obtain

$$\delta Z^M E_M{}^A(Z) = \delta Z_0^M E_M{}^A + (\delta y^A + y^D \delta Z_0^M E_M{}^E \Omega_{ED}{}^A) + \delta Z_0^M E_M{}^C y^B T_{BC}{}^A$$

$$+\frac{1}{2}(\delta y^C + y^D \delta Z_0^M E_M{}^E \Omega_{ED}{}^C) y^B T_{BC}{}^A + \frac{1}{2}\delta Z_0^M E_M{}^D y^E T_{ED}{}^C y^B T_{BC}{}^A$$
$$+\frac{1}{2}\delta Z_0^M E_M{}^C y^B y^D D_D T_{BC}{}^A + \frac{1}{2} y^D \delta Z_0^M E_M{}^C y^B R_{BCD}{}^A + o(y^3). \quad (43)$$

Once the left-hand side of this equation is known and once one specifies δZ_0^M this equation can be perturbatively solved for δy^A. For κ-transformations we have

$$\delta_\kappa Z_0^M E_M{}^A(Z_0) = \Delta^A(Z_0) \quad (44a)$$
$$\delta_\kappa Z^M E_M{}^A(Z) = \Delta^A(Z) = \Delta^A(Z_0) + y^B D_B \Delta^A(Z_0) + o(y^2) \quad (44b)$$

where $\Delta^a = 0$ and Δ^α is given in Eq. (14). Notice that δy^A appears in (43) always in the combination $\delta y^A + y^B \Delta^\gamma \Omega_{\gamma B}{}^A$ and that all other terms are Lorentz-covariant. Therefore we can solve this equation perturbatively to get a Lorentz-covariant expression for this combination. With the aid of (44) we obtain the κ-transformations for y^A which, together with (44a), leave the expanded action invariant:

$$\delta_\kappa y^a = -y^c \Delta^\gamma \Omega_{\gamma c}{}^a - \Delta^\gamma y^B T_{B\gamma}{}^a + o(y^2) \quad (45a)$$
$$\delta_\kappa y^\alpha = -y^\beta \Delta^\gamma \Omega_{\gamma\beta}{}^\alpha - \Delta^\gamma y^B T_{B\gamma}{}^\alpha + y^B \mathcal{D}_B \Delta^\alpha + o(y^2). \quad (45b)$$

The $o(y^2)$ terms are all Lorentz-covariant. So we see that on the y's κ-transformations can be considered as a combination of a field-dependent Lorentz-transformation, with parameter $L_a{}^b \equiv \Delta^\gamma \Omega_{\gamma a}{}^b$, and an "intrinsic" Lorentz-preserving κ-transformation.

The BRS transformations on Ψ can be obtained in a similar way. We write (see (32c)):

$$\Big(\partial_i - A_i(Z)\Big)\psi = e^\Lambda \Big[(\partial_i - A_i(Z_0))(\psi_0 + \Psi) - \Big(V_i{}^A y^B F_{BA} +$$
$$+\frac{1}{2}\left(D_i y^A + V_i{}^C y^D T_{DC}{}^A\right) y^B F_{BA} + \frac{1}{2} V_i{}^A y^B y^C \mathcal{D}_C F_{BA} + o(y^3)\Big)(\psi_0 + \Psi)\Big] \quad (46)$$

For generic variations $\delta\psi$, $\delta\psi_0$, $\delta\Psi$, δZ^M, δZ_0^M, δy^A we get therefore

$$\delta\psi - \delta Z^M A_M \psi = e^\Lambda \Big[\delta\psi_0 - \delta Z_0^M A_M \psi_0 + \delta\Psi - \delta Z_0^M A_M \Psi -$$
$$-\Big(\delta Z_0^M E_M{}^A y^B F_{BA} + \frac{1}{2}(\delta y^A + y^E \delta Z_0^M \Omega_{ME}{}^A + \delta Z_0^M E_M{}^C y^D T_{DC}{}^A) y^B F_{BA}$$
$$+\frac{1}{2}\delta Z_0^M E_M{}^A y^B y^C \mathcal{D}_C F_{BA} + o(y^3)\Big)(\psi_0 + \Psi)\Big]. \quad (47)$$

If we apply this formula to κ-transformations we see that the l.h.s. vanishes identically. On ψ_0 we impose its classical κ-transformation

$$\delta_\kappa \psi_0 = C\psi_0, \tag{48}$$

$\delta_\kappa Z_0^M$ is known and $\delta_\kappa y^A$ has been determined above. Notice that again only the Lorentz-covariant combination $\delta_\kappa y^A + y^B \Delta^\gamma \Omega_{\gamma B}{}^A$ appears. Therefore (47) determines the κ-transformation of the quantum heterotic fermions:

$$\begin{aligned}\delta_\kappa \Psi = C\Psi + \Big(&\Delta^\alpha y^b F_{b\alpha} + \frac{1}{2} y^C D_C \Delta^\alpha y^b F_{b\alpha} \\ &+ \frac{1}{2}\Delta^\alpha y^b y^C \mathcal{D}_C F_{b\alpha} + o(y^3)\Big)(\psi_0 + \Psi).\end{aligned} \tag{49}$$

Also in this case we see that on the quantum fields Ψ the κ-transformations act as a field-dependent gauge transformation, with transformation parameter $C = \Delta^\alpha A_\alpha$, plus an "intrinsic" gauge and Lorentz covariant κ-transformation.

The BRS transformation of the quantum zweibeins $h_\pm{}^i$ can be obtained by expanding (13c) and (13d) and demanding again that $e_{0\pm}{}^i$ transforms "classically". The κ-transformations are given by

$$\delta_\kappa h_-{}^i = 0 \tag{50a}$$

$$\begin{aligned}\delta_\kappa h_+{}^i = &-4e_{0-}{}^i \left(D_+ y^\varepsilon + V_+{}^B y^C T_{CB}{}^\varepsilon + \frac{1}{2}\psi_0 y^A \mathcal{D}_A \chi^\varepsilon \psi_0\right)\kappa_{+\varepsilon} - \\ &- 4h_-{}^i \left(V_+{}^\varepsilon - \frac{1}{2}\psi_0 \chi^\varepsilon \psi_0\right)\kappa_{+\varepsilon} + o(y^2).\end{aligned} \tag{50b}$$

Now we have to be more specific about our gauge-fixing choice.

To fix world-sheet diffeomorphisms, Weyl and Lorentz invariance we shall impose the condition

$$h_\pm{}^i = 0 \tag{51}$$

on the zweibeins quantum fields.

For what concerns κ-invariance, until now no $D = 10$ Lorentz-preserving quantization procedure is known. Therefore, as unpleasant as it may be, we are obliged to resort to a non-covariant gauge-fixing [12]. Consequently we shall fix

κ-symmetry by introducing two light-like ten-dimensional constant vectors m^a, n^a satisfying

$$\begin{aligned} m^a n_a &= \frac{1}{2} \\ m^a m_a &= 0 = n^a n_a \end{aligned} \tag{52}$$

such that the matrices $\not{m} \equiv m_a \Gamma^a$, $\not{n} \equiv n_a \Gamma^a$ can be used to project $SO(10)$ spinors down to $SO(8)$ spinors. We impose

$$\not{n}_{\alpha\beta} y^\beta = 0 \tag{53}$$

and restrict the background-connection $\Omega_{ia}{}^b(Z_0)$ according to

$$\Omega_{iab} n^b = 0 = \Omega_{iab} m^b \tag{54}$$

such that the covariant derivative preserves (53)

$$\not{n}_{\alpha\beta} D_i y^\beta = 0. \tag{55}$$

As a consequence of (54) we will get an $SO(8)$-invariant effective action and can finally use this residual $SO(8)$ invariance to covariantize our results back to $SO(10)$. This procedure supposes that in principle an $SO(10)$ Lorentz-covariant quantization scheme is available.

As for the huge series of secondary symmetries which arise due to the (infinite) reducibility of κ-symmetry, they will be fixed by imposing on the quantum fields of the κ-ghosts, antighosts, LN fields and secondary ghosts conditions like Eq. (53) involving alternatively the constant vectors m^a and n^a [12]. These conditions together with the relevant field equations imply that the whole chain of κ-ghosts do not propagate in our gauge and can be disregarded at the quantum level.

However the ghosts and antighosts of diffeomorphisms do propagate and in a complete treatment they should be taken into account carefully. Yet in this paper we are interested only on the κ-anomaly (at one loop) and the diffeomorphisms ghosts are expected not to contribute with this respect.

We end this section by giving the normal-coordinate-expanded lagrangian at second order in the quantum fields which is needed for our one-loop computations. In performing the expansion along the quantum variables we need also

the relations (7) stemming from the solution of the Bianchi identities with the constraints (6). We get (for $h^i_\pm = 0$)

$$
\begin{aligned}
L_2 = \sqrt{g}\Big[& y^\alpha V_-{}^a (\Gamma_a)_{\alpha\beta} D_+ y^\beta - \frac{1}{2} D_- y_a D_+ y^a - 2 D_- y^a V_+{}^\beta y^\alpha (\Gamma_a)_{\alpha\beta} \\
& + 2 V_-{}^a V_+{}^b y^\alpha T_{cb}{}^\varepsilon (\Gamma_a)_{\varepsilon\alpha} y^c - \frac{1}{4} V_-{}^a V_+{}^\alpha y_d y^\gamma (\Gamma_a)_{\alpha\varphi} (\Gamma_{bc})^\varphi{}_\gamma T^{bcd} \\
& - \frac{1}{2} V_-{}^a V_+{}^b y^d y^c R_{dacb} + \frac{1}{2} D_- y^a V_+{}^b T_{gba} y^g - V_-{}^\beta V_+{}^b T_{cb}{}^\alpha y^c y^g (\Gamma_g)_{\alpha\beta} \\
& + \frac{1}{4} V_-^a V_+^b y^\delta y^\gamma (\Gamma_{acd})_{\delta\gamma} T^{cd}{}_b + V_-^a V_+{}^\alpha T_{\delta\alpha}{}^\beta (\Gamma_a)_{\beta\gamma} y^\delta y^\gamma - 2 V_-{}^\alpha V_+{}^\beta (\Gamma^g)_{\alpha\delta} (\Gamma_g)_{\beta\gamma} y^\gamma y^\delta \\
& + \frac{1}{2} \Psi \mathcal{D}_- \Psi \Big] + o(\Psi y) + o(\psi_0^2 y^2).
\end{aligned} \tag{56}
$$

The $o(\Psi y) + o(\psi_0^2 y^2)$ terms will not enter our calculations so we did not write them explicitly.

In the next section we will start doing one-loop computations.

5 A non standard derivation of the Yang-Mills anomaly and the related κ-anomaly

The normal-coordinate expanded lagrangian is also invariant under Yang-Mills and Lorentz gauge transformations involving both the background fields and the quantum fields. The Yang-Mills transformations are

$$
\begin{aligned}
\delta_G A_i &= \mathcal{D}_i \mathcal{C} \equiv \partial_i \mathcal{C} + \mathcal{C} A_i - A_i \mathcal{C} \\
\delta_G \Psi &= \frac{1+\gamma_3}{2} \mathcal{C} \Psi,
\end{aligned} \tag{57}
$$

where we have reintroduced the two-component notation for the quantum heterotic fermions, and the local Lorentz transformations are

$$
\begin{aligned}
\delta_L \Omega_{iA}{}^B &= D_i \mathcal{L}_A{}^B \equiv \partial_i \mathcal{L}_A{}^B + \mathcal{L}_A{}^C \Omega_{iC}{}^B - \Omega_{iA}{}^C \mathcal{L}_C{}^B \\
\delta_L y^A &= -y^B \mathcal{L}_B{}^A \\
\delta_L V_i{}^A &= -V_i{}^B \mathcal{L}_B{}^A \\
\delta_L T_{A\cdots}{}^{B\cdots} &= \mathcal{L}_A{}^C T_{C\cdots}{}^{B\cdots} - T_{A\cdots}{}^{C\cdots} \mathcal{L}_C{}^B + \cdots
\end{aligned} \tag{58}
$$

where $\mathcal{C}$ is a local Lie algebra-valued parameter, $\mathcal{C} = \mathcal{C}^I T^I$ and $\mathcal{L}$ is a Lorentz-valued parameter, ${\mathcal{L}_a}^\alpha = {\mathcal{L}_\alpha}^a = 0$, ${\mathcal{L}_\alpha}^\beta = \frac{1}{4}(\Gamma^{ab})_\alpha{}^\beta \mathcal{L}_{ab}$; $\mathcal{D}_i$ and D_i are the gauge and Lorentz induced covariant derivatives, respectively and $T_{A\ldots}{}^{B\ldots}$ is any Lorentz tensor.

As a consequence it is meaningful to speak of the anomalies of these symmetries. The consideration of these Yang-Mills and Lorentz anomalies is a useful tool to discuss the κ-anomalies associated to Yang-Mills and Lorentz Chern-Simons forms, on which we are interested in this paper.

In this section we want to compute the by now well understood gauge-anomaly of the Green-Schwarz sigma model in dimensional regularization by a non standard method [27]. This rederivation of the gauge anomaly will clarify also some aspect of the appearance of the κ-anomaly associated to the Yang-Mills Chern-Simons form and guide us also in the derivation of the Lorentz anomaly and the κ-anomaly associated to the Lorentz Chern-Simons form.

Our computational method is based on the following rather general consideration. Consider an action $I[\chi, \phi_0]$ which depends on a set of external fields ϕ_0, and on a set of quantum fields χ over which we are going to perform a path integration. Let us moreover assume that the action is at the classical level invariant under a set of transformations $\delta\phi_0$, $\delta\chi$ with associated BRS charge Ω

$$\Omega I = 0. \tag{59}$$

Ω is a nilpotent operator if the algebra of the symmetry transformations is closed, but when the algebra closes only on-shell (open algebra), as is the case of κ-transformations, Ω is nilpotent only on-shell. In the Batalin-Vilkovisky approach Eq. (59) is replaced by the master equation (25) for the extended action.

The Slavnov operator $(S, \cdot)$ is nilpotent in all cases once S satisfies the master equation. When the action I (the extended action S) is regularized dimensionally, going in $d = 2 + \epsilon$ dimensions, one gets an action I_ϵ (S_ϵ) which is no longer invariant (no longer satisfies the master equations) if the regularization breaks the symmetry

$$(S_\epsilon, S_\epsilon) = \mathcal{Q}_\epsilon = \epsilon \mathcal{R}_\epsilon \tag{60}$$

or

$$\Omega I_\epsilon = Q_\epsilon = \epsilon R_\epsilon \tag{61}$$

where $R_\epsilon = \mathcal{R}_\epsilon\Big|_{\chi^*=\phi_0^*=0}$. If $\epsilon \to 0$ $I_\epsilon \to I$ ($S_\epsilon \to S$) and $Q_\epsilon \to 0$ ($\mathcal{Q}_\epsilon \to 0$).

It is convenient to define an action S_ϵ^η, introducing an anticommuting constant parameter η which at the end will be set to zero, according to

$$S_\epsilon^\eta = S_\epsilon + \eta \mathcal{R}_\epsilon \tag{62}$$

and Eq. (60) becomes

$$(S_\epsilon^\eta, S_\epsilon^\eta) = \epsilon \frac{\delta S_\epsilon^\eta}{\delta \eta}. \tag{63}$$

The effective action $\mathit{\Gamma}_\epsilon^\eta$ no longer satisfies the Slavnov-Taylor identity (35) which is now replaced by

$$(\mathit{\Gamma}_\epsilon^\eta, \mathit{\Gamma}_\epsilon^\eta) = \epsilon \frac{\delta \mathit{\Gamma}_\epsilon^\eta}{\delta \eta} \tag{64}$$

or

$$\Omega \Gamma_\epsilon^\eta = \epsilon \frac{\delta \Gamma_\epsilon^\eta}{\delta \eta} \tag{65}$$

where $\Gamma_\epsilon^\eta = \mathit{\Gamma}_\epsilon^\eta\Big|_{\phi_0^*=0}$.

Due to the analyticity of the dimensional regularization at first order in α ($\alpha = 2\pi\alpha'$ plays here the role of Planck's constant) i.e. at one loop, we can make the following expansion in η

$$\Gamma_\epsilon^\eta = I_\epsilon^\eta + \alpha \left(\left(\Gamma_1 + \frac{1}{\epsilon}\Gamma_0 \right) + \eta \left(\Delta_1 + \frac{1}{\epsilon}\Delta_0 \right) \right) \tag{66}$$

where Γ_1, Δ_1 are finite and Γ_0, Δ_0 parametrize the divergent local contributions to the effective action. Putting this into (65) and setting then $\eta = 0$ we get for the regularized physical effective action

$$\Omega \Gamma_\epsilon = \epsilon \left(R_\epsilon + \alpha(\Delta_1 + \frac{1}{\epsilon}\Delta_0) \right), \tag{67}$$

and for $\epsilon \to 0$

$$\Omega \Gamma = \alpha \Delta_0. \tag{68}$$

The Wess-Zumino consistency condition is then

$$\Omega \Delta_0 = 0. \tag{69}$$

If Δ_0 cannot be written as the Ω-variation of a local action it constitutes an anomaly.

Here we note that, thanks to (68) and (66) 1) The anomaly Δ_0 is local and finite; 2) the divergent part of the effective action is BRS invariant. What we learned from these considerations, taking a look at (66), is that the anomaly can be computed by inserting the "anomalous vertex" R_ϵ once in all one-loop diagrams and keeping the $1/\epsilon$-divergent contributions or, alternatively, by inserting Q_ϵ and taking the limit for $\epsilon \to 0$. With respect to the traditional perturbative procedure where one computes first the effective action via Feynman diagrams and then makes a variation we reversed the order: we make first a variation of the regularized action and then compute Feynman diagrams. One advantage of this procedure is that one never meets non-local terms which arise typically in the traditional procedure where the anomaly stems from diagrams with different numbers of external legs, which have to be combined with non-local contributions, as is for example the case for non-abelian ABBJ anomalies in any even dimension.

Let us now apply this procedure to compute the Yang-Mills anomaly coming from the heterotic sector. For a proper definition of the propagator for the quantum heterotic fermions we have to augment the action

$$I_H = \frac{1}{2}\int d^2\sigma\sqrt{g}e_p{}^i\overline{\Psi}\gamma^p\frac{1+\gamma_3}{2}\mathcal{D}_i\Psi \tag{70}$$

by the decoupled term

$$I'_H = \frac{1}{2}\int d^2\sigma\overline{\Psi}\gamma^p\frac{1-\gamma_3}{2}\partial_p\Psi \tag{71}$$

which is trivially invariant under all local symmetries since we choose $\frac{1-\gamma_3}{2}\Psi$ to be a singlet under all transformations. The dependence on the determinant g of the heterotic fermions terms (70) and (71) is fictitious in that g can be eliminated by rescaling the heterotic fermion fields Ψ. We use this freedom to write the heterotic fermions action as

$$I_H = \frac{1}{2}\int d^2\sigma\sqrt{\grave{g}}\grave{e}_p{}^i\overline{\Psi}\gamma^p\left(\partial_i - \frac{1+\gamma_3}{2}A_i\right)\Psi. \tag{72}$$

where we have introduced the left-accented zweibeins $\grave{e}_+{}^i = \delta^i_+$, $\grave{e}_-{}^i = e_-{}^i$. Later we will use the right-accented zweibeins $\acute{e}_+{}^i = e_+{}^i$, $\acute{e}_-{}^i = \delta_-{}^i$.

We have now to dimensionally extend this action; for that we shall follow the t'Hooft-Veltman recipe as formulated by Breitenlohner and Maison [31]. We go to $D = 2 + \epsilon$ dimensions keeping consistently γ^3 strictly in two dimensions and splitting a D-dimensional vector index i as $i = (\bar{\imath}, \hat{\imath})$ where $\bar{\imath}$ stays strictly in 2 and $\hat{\imath}$ denotes the extra ϵ dimensions. A similar splitting is adopted for the flat indices $p = (\overline{p}, \hat{p})$. The Dirac algebra becomes then [31]

$$\begin{aligned} \{\gamma^p, \gamma^q\} &= 2\eta^{pq} \\ \{\overline{\gamma}^p, \gamma^3\} &= 0 \\ \left[\hat{\gamma}^p, \gamma^3\right] &= 0. \end{aligned} \tag{73}$$

We compute the gauge anomaly for the classical flat metric $\grave{g}^{ij} = \eta^{ij}$ restoring the metric $\grave{g}^{ij}$ at the end.

Performing now the transformations given in (57) we compute the anomalous vertex associated to the dimensionally extended action gotten from (72) in a flat metric to be

$$\Omega I_H^\epsilon = \frac{1}{2} \int d^D\sigma \overline{\Psi} \mathcal{C} \hat{\gamma}^i \gamma^3 \hat{\mathcal{D}}_i \Psi \tag{74}$$

where $\hat{\mathcal{D}}_i = \hat{\partial}_i - \frac{1+\gamma_3}{2} \hat{A}_i$. Due to the fact that the connection A_i is an external field which lives strictly in two dimensions we get for the anomalous vertex

$$\epsilon R_\epsilon = Q_\epsilon = \frac{1}{2} \int d^D\sigma \, \overline{\Psi} \mathcal{C} \hat{\gamma}^i \gamma^3 \hat{\partial}_i \Psi. \tag{75}$$

Q_ϵ contains as external fields only the ghost field $\mathcal{C} = \mathcal{C}^I T^I$ which is attached to a fermion line.

The Feynman rules are the usual ones

$$\Psi \text{ propagator} \qquad \frac{i\alpha}{\not{k}} \delta^{rs} \tag{76a}$$

$$\Psi\text{-}\overline{\Psi}\text{-}A \text{ gauge vertex} \qquad \frac{1}{\alpha} \gamma^i \frac{1+\gamma^3}{2} T^I. \tag{76b}$$

The Feynman rule associated to the anomalous vertex (75) is given by

$$\Psi\text{-}\Psi\text{-}\mathcal{C} \text{ anomalous vertex} \qquad \frac{i}{\alpha} \frac{\hat{\not{k}} - \hat{\not{k}}'}{2} \gamma^3 T^J \tag{76c}$$

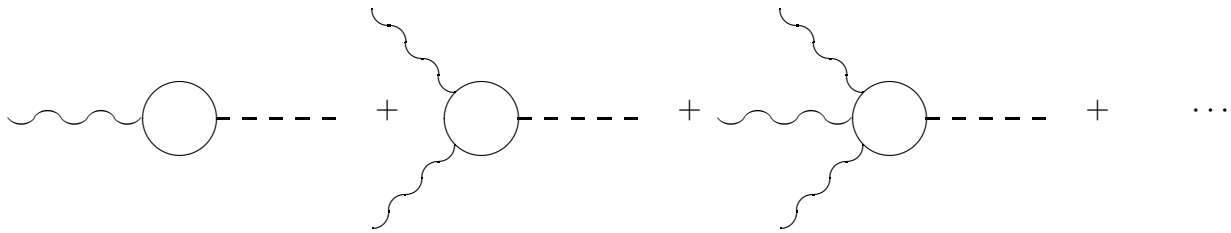

Figure 1: Fermionic graphs contributing to the one-loop anomaly.

where J is the gauge index carried by the external ghost field and k and k' are the incoming and outgoing momenta of the fermions. The Feynman graphs at one loop with the insertion of *one* anomalous vertex of the type (76c) are indicated in Fig. 1.

Let us compute the anomaly arising from the first diagram in that figure; it contains one external gauge field A^I and an external ghost C^J associated to (76c) while in the loop are circulating fermions. Keeping the external momenta strictly in two dimensions it is given by

$$\frac{i}{\alpha}\mathcal{A}_j^{IJ}(p) = \frac{i}{2}\,\mathrm{tr}(T^I T^J)\int \frac{d^D k}{(2\pi)^D}\,\mathrm{tr}\left(\hat{\not{k}}\gamma_3 \frac{1}{\not{k}}\gamma_j \frac{1+\gamma_3}{2}\frac{1}{\not{k}-\not{p}}\right). \tag{77}$$

The integral over k is ultraviolet (logarithmically) divergent, on the other hand $\hat{k}$ is of order ϵ so that the result is expected to be finite. A careful calculation gives in fact, in the limit $\epsilon \to 0$:

$$\mathcal{A}_j^{IJ} = \frac{\alpha}{8\pi}\,\mathrm{tr}(T^I T^J)(\eta_{mj} - \varepsilon_{mj})(ip^m). \tag{78}$$

Upon adding the external legs, A^I and $\mathcal{C}^J$, and transforming back to configuration space one gets for the gauge anomaly

$$\mathcal{A}'_G = -\frac{\alpha}{8\pi}\int d^2\sigma\,\mathrm{tr}(\mathcal{C}\partial_+ A_-). \tag{79}$$

If we go on to consider the diagrams with n external legs in Fig. 1, we may notice that the integration over the loop-momentum behaves for large k as

$$\sim \int^{\Lambda} \frac{d^D k}{k^{n+1}}\hat{k} \sim \Lambda^{2-n}. \tag{80}$$

Now for $n \geq 3$ the integral over k in (80) is surely convergent and due to the presence of $\hat{k}$ in the numerator, as $\epsilon \to 0$ the amplitude vanishes. So there is no

contribution to the anomaly coming from all the diagrams in Fig. 1 with three or more external gauge fields. The unique case to be considered remains the diagram with the insertion of *two* gauge fields. In this case one gets a logarithmically divergent integral in (80) and for $\epsilon \to 0$ one can set the external momenta to zero. For the second diagram in Fig. 1 one gets

$$\mathcal{A}_2 = \int \frac{d^D k}{(k^2)^3} \mathcal{J}(k^3, \hat{k}). \tag{81}$$

The function $\mathcal{J}(k^3, \hat{k})$ is written explicitly in the appendix. It is constituted by a trace over γ-matrices containing three powers of momenta in D dimensions and one $\hat{k}$ which lives in ϵ dimensions. A careful analysis of this trace of γ-matrices reveals, however, that actually $\mathcal{A}_2 = 0$ identically (see appendix A). As a result also this diagram vanishes and we are left with the anomaly computed in (79) .

Restoring the left-accented metric we have

$$\mathcal{A}'_G = -\frac{\alpha}{8\pi} \int d^2\sigma \sqrt{\grave{g}}\, \mathrm{tr}(\mathcal{C} \grave{D}_+ A_-), \tag{82}$$

which is not invariant under diffeomorphisms. Here we defined the Weyl and $d = 2$ local Lorentz covariant derivatives for a generic zweibein $e_\pm{}^i$

$$D_\pm = \partial_\pm + \frac{1}{\sqrt{g}} \partial_j (\sqrt{g} e_\pm{}^j). \tag{83}$$

We can get a diff-invariant form of the anomaly by adding a local term (in dimensional regularization the effective action is always defined modulo local terms); we redefine the effective action according to

$$\Gamma_H = \Gamma'_H - \frac{\alpha}{16\pi} \int d^2\sigma \sqrt{\grave{g}}\, \mathrm{tr}(\grave{A}_+ A_-) \tag{84}$$

to get the metric-independent gauge anomaly

$$\mathcal{A}_G = \frac{\alpha}{8\pi} \int d^2\sigma \varepsilon^{ij}\, \mathrm{tr}(\mathcal{C} \partial_i A_j). \tag{85}$$

Clearly this anomaly can also be deduced directly by integrating (72) over the fermions and computing the A-A contribution to the effective action (Fig. 2). For a flat metric, with our (dimensional) regularization, one gets

$$\begin{aligned} \Gamma'_H &= -\frac{\alpha}{16\pi} \int d^2\sigma (\eta^{ij} - \varepsilon^{ij})(\eta^{mn} - \varepsilon^{mn})\, \mathrm{tr}\left(\partial_i A_j \frac{1}{\Box} \partial_m A_n\right) \\ &= -\frac{\alpha}{16\pi} \int d^2\sigma\, \mathrm{tr}\left(\partial_+ A_- \frac{1}{\Box} \partial_+ A_-\right). \end{aligned} \tag{86}$$

We can restore the left-accented metric to obtain

$$\Gamma'_H = -\frac{\alpha}{16\pi}\int d^2\sigma\sqrt{\grave{g}}\,\mathrm{tr}\left(\grave{D}_+A_-\frac{1}{\Box_{\grave{g}}}\grave{D}_+A_-\right), \tag{87}$$

where, for a generic metric g_{ij},

$$\Box_g \equiv \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) = D_+\partial_- = D_-\partial_+. \tag{88}$$

As it stands, (87) suffers a diffeomorphisms anomaly which is however trivial and can be eliminated by redefining Γ'_H as in (84) to get finally:

$$\begin{aligned}\Gamma_H &= -\frac{\alpha}{16\pi}\int d^2\sigma\sqrt{\grave{g}}\,\mathrm{tr}\left(\grave{D}_+A_-\frac{1}{\Box_{\grave{g}}}(\grave{D}_+A_- - D_-\grave{A}_+)\right)\\ &= -\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(A_-\frac{1}{D_-}\frac{\varepsilon^{ij}\partial_iA_j}{\sqrt{g}}\right).\end{aligned} \tag{89}$$

It is not difficult to convince ourselves that actually the determinant $\sqrt{g}$ scales away in (89) and therefore we were allowed to replace $\sqrt{\grave{g}}$ with $\sqrt{g}$. Varying this action according to (57) we get

$$\begin{aligned}\delta_G\Gamma_H = &\frac{\alpha}{8\pi}\int d^2\sigma\varepsilon^{ij}\,\mathrm{tr}(\mathcal{C}\partial_iA_j) -\\ &-\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(D_+[\mathcal{C},A_-]\frac{1}{\Box_g}D_+A_-\right).\end{aligned} \tag{90}$$

The first term in (90) is local and corresponds to the anomaly (85) while the second term is non-local and is clearly spurious in the sense that it gets cancelled by a corresponding term in the variation of Γ_3, see the second diagram in Fig. 2 with three external gauge fields A_i. Now, also $\delta(\Gamma_H+\Gamma_3)$ contains, apart from $\mathcal{A}_G$, non-local terms which are cancelled by $\delta\Gamma_4$ and so on. These cumbersome linked cancellations which are due to the non abelian nature of the Yang-Mills gauge fields are elegantly avoided by the non-standard method we employed above, because in that case the diagrams with two or more external gauge fields do simply not contribute. Actually, our non-standard derivation of the gauge anomaly constitutes a proof of these linked cancellations.

We turn now to the derivation of the κ-anomaly in the Yang-Mills sector coming from the functional integration over the heterotic fermions of (72). The

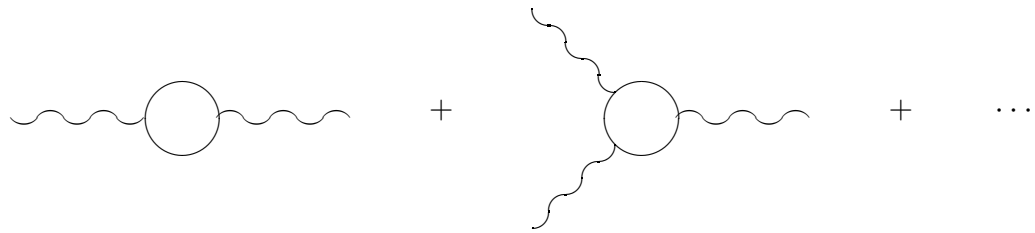

Figure 2: Fermionic graphs contributing to the one-loop effective action.

κ-transformations of the fields comparing in (72) are given by (here we use again the two-component notation)

$$\delta_\kappa A_i = \mathcal{D}_i C + F_i \tag{91a}$$

$$\begin{aligned}\delta_\kappa \Psi = \frac{1+\gamma_3}{2}\Big[& C\Psi + \Big(\Delta^\alpha y^b F_{b\alpha} + \frac{1}{2} y^C D_C \Delta^\alpha y^b F_{b\alpha} \\ & + \frac{1}{2}\Delta^\alpha y^b y^C \mathcal{D}_C F_{b\alpha} + o(y^3)\Big)(\psi_0 + \Psi)\Big].\end{aligned} \tag{91b}$$

$$\delta_\kappa \grave{e}_p{}^i = 0$$

where we recall that $C = \Delta^\alpha A_\alpha$. As we observed already, the κ-transformations act like a field-dependent gauge transformation with parameter C plus an intrinsic κ-transformation. Notice that $F_- = 0$, see (38), and that only A_- is coupled to the heterotic fermions in (72).

The field-dependent gauge transformation gives therefore rise to an anomalous κ-vertex which is given by (75) where $\mathcal{C}$ has to be substituted by C. The related κ-anomaly can then be computed in complete analogy to the gauge anomaly (82) and one gets

$$\mathcal{A}_G^\kappa = -\frac{\alpha}{8\pi}\int d^2\sigma \sqrt{\grave{g}}\, \mathrm{tr}\left(C \grave{D}_+ A_-\right). \tag{92}$$

Again this anomaly is not diff-invariant and we add to the effective action the same cocycle as in (84) to obtain the diff-invariant κ-anomaly,

$$\begin{aligned}\mathcal{A}_G^\kappa &= \frac{\alpha}{8\pi}\int d^2\sigma \varepsilon^{ij}\, \mathrm{tr}(C\partial_i A_j + F_i A_j) \\ &= -\frac{\alpha}{16\pi}\int d^2\sigma \varepsilon^{ij} V_i{}^A V_j{}^B \Delta^\gamma (\omega_{3YM})_{\gamma BA}.\end{aligned} \tag{93}$$

The first line in (93) stems from (38) while the second line involves the definition of the Yang-Mills Chern-Simons form ω_{3YM} given in (8). The intrinsic

κ-transformations are expected not to contribute to the κ-anomaly at one loop since, as we will see in section VIII, (93) satisfies already the Wess-Zumino consistency condition.

Taking a look at (17) one realizes that $\mathcal{A}_G^\kappa$ can be eliminated [25] by imposing the constraints, which are imposed on $W = dB$ in (6) at the classical level, on the three-superform H defined as

$$H = dB + \frac{\alpha}{8\pi}\omega_{3YM}. \tag{94}$$

This relation then requires that B has to transform anomalously under gauge-transformations according to

$$\delta_G B = -\frac{\alpha}{8\pi}\,\mathrm{tr}(\mathcal{C}dA) \tag{95}$$

because $\delta_G\omega_{3YM} = d(\mathcal{C}dA)$. Then, taking (95) into account, the gauge transformation of the action (10) cancels the gauge anomaly (85), as is well known.

The Bianchi identity associated to (94) is

$$dH = \frac{\alpha}{8\pi}\,\mathrm{tr}(FF), \tag{96}$$

it can be consistently solved in superspace [28], and it gives rise to the Chapline-Manton theory [29], i.e. constitutes the minimally coupled SUGRA-SYM theory in ten dimensions. Eq. (96) coincides with the result of Ref. [3] by taking into account that our H differs from the one used in that reference by a factor of two.

6 The Lorentz anomaly

In this section we want to derive the Lorentz anomaly of the sigma model with the same technique we used in the previous section to derive the gauge anomaly. A Lorentz anomaly is expected to appear due to the chiral coupling of the anticommuting y^α to the induced Lorentz connection $\Omega_{i\alpha}{}^\beta \equiv \frac{1}{4}\Omega_{iab}(\Gamma^{ab})_\alpha{}^\beta$, $\Omega_{ia}{}^b \equiv V_i^C\Omega_{Ca}{}^b$, in the first term in (56), through the covariant derivative

$D_j y^\beta \equiv \partial_j y^\beta - \Omega_j{}^\beta{}_\gamma y^\gamma$. This term is invariant under the Lorentz transformations

$$\delta_L y^\alpha = \mathcal{L}^\alpha{}_\beta y^\beta \tag{97a}$$

$$\delta_L \Omega_i{}^\beta{}_\alpha = \partial_i \mathcal{L}^\beta{}_\alpha + \mathcal{L}^\beta{}_\gamma \Omega_i{}^\gamma{}_\alpha - \Omega_i{}^\beta{}_\gamma \mathcal{L}^\gamma{}_\alpha \tag{97b}$$

$$\mathcal{L}^\beta{}_\alpha = \frac{1}{4}(\Gamma_{ab})^\beta{}_\alpha \mathcal{L}^{ab}.$$

Since there exists up to now no $SO(10)$ Lorentz-covariant quantization of the theory we limit ourselves to derive the Lorentz anomaly under $SO(8)$ transformations, i.e. such that

$$m_a \mathcal{L}^{ab} = n_a \mathcal{L}^{ab} = 0. \tag{98}$$

To get a canonical kinetic term for the worldsheet scalars y^α they have to be transformed to worldsheet Majorana-Weyl fermions [24]. This can be achieved by rescaling the y^α by an $SO(8)$ invariant quantity

$$y^\alpha = \frac{1}{\sqrt{4n_-}} y_u^\alpha, \tag{99}$$

where $n_- = V_-{}^a n_a$, and by introducing a worldsheet Majorana spinor as

$$Y^\alpha = \begin{pmatrix} y_u^\alpha \\ y_d^\alpha \end{pmatrix} \tag{100}$$

whose bottom component $1/2(1+\gamma^3)Y^\alpha = \begin{pmatrix} 0 \\ y_d^\alpha \end{pmatrix}$ is decoupled from the theory.

Then the first term in (56) can be rewritten as follows:

$$\sqrt{g} y^\alpha \not{V}_{-\alpha\beta} D_+ y^\beta = \frac{1}{4n_-} \sqrt{g} \overline{Y}^\alpha e_p{}^i \gamma^p \frac{1-\gamma_3}{2} \not{V}_{-\alpha\beta} D_i Y^\beta. \tag{101}$$

To complete the action of the Majorana-Weyl fermions Y^α we have to add the decoupled kinetic term of the y_d^α; in analogy with the discussion on the heterotic fermions kinetic term we get

$$I_F = \frac{1}{2} \int d^2\sigma \sqrt{\acute{g}} \left(\frac{1}{2n_-} \overline{Y}^\alpha \acute{e}_p{}^i \gamma^p \frac{1-\gamma_3}{2} \not{V}_{-\alpha\beta} D_i Y^\beta + \acute{e}_p{}^i Y^\alpha \gamma^p \frac{1+\gamma_3}{2} \not{n}_{\alpha\beta} \partial_i Y^\beta \right). \tag{102}$$

where now we use the right-accented zweibeins $\acute{e}_+{}^i = e_+{}^i$, $\acute{e}_-{}^i = \delta^i_-$. I_F is invariant under $SO(8)$ local transformations, i.e. (97b) and

$$\delta_L Y^\alpha = \frac{1-\gamma_3}{2}\mathcal{L}^\alpha{}_\beta Y^\beta. \tag{103}$$

Now we can proceed along the lines of the preceding section to compute the Lorentz anomaly. We use dimensional regularization to extend I_F to I_F^ϵ in precisely the same manner as we did in the preceding section for the heterotic fermions and compute the anomalous vertex associated to (102) for a flat metric $\acute{g}^{ij} = \eta^{ij}$. We get

$$\Omega_L I_F^\epsilon = -\frac{1}{2}\int d^D\sigma \overline{Y}^\alpha \mathcal{L}_\alpha{}^\beta \hat{\gamma}^i \gamma^3 \left(\frac{V_-{}^a}{2n_-}(\Gamma_a)_{\beta\gamma}\frac{1-\gamma_3}{2}D_i + \not{n}_{\beta\gamma}\frac{1+\gamma_3}{2}\partial_i\right) Y^\gamma. \tag{104}$$

To compute the anomaly we enforce now the κ gauge-fixing (53) which becomes

$$\frac{1-\gamma_3}{2}(\not{n}Y)_\alpha = 0. \tag{105}$$

To do this we insert the identity $\not{n}\not{n} + \not{n}\not{n} = 1$ in (102) and in (104), to get

$$\tilde{I}_F^\epsilon = \frac{1}{2}\int d^D\sigma \overline{Y}^\alpha \gamma^i \left(\partial_i \not{n}_{\alpha\beta} - \frac{1-\gamma_3}{2}\Omega_{i\alpha}{}^\gamma \not{n}_{\gamma\beta}\right) Y^\beta \tag{106}$$

$$\epsilon R_\epsilon = Q_\epsilon = -\frac{1}{2}\int d^D\sigma Y^\alpha \mathcal{L}_\alpha{}^\beta \hat{\gamma}^i \gamma^3 \not{n}_{\beta\gamma}\hat{\partial}_i Y^\gamma. \tag{107}$$

We used the fact that $[\mathcal{L}, \not{n}] = 0 = [\Omega_i, \not{n}]$ and that $\Omega_{i\alpha}{}^\beta$ lives strictly in two dimensions.

Now we can use the formal analogy between (106), (107) and (72), (75) to compute the anomaly. From (106) we deduce the Feynman rules

$$Y^\alpha \text{ propagator} \qquad \frac{i\alpha}{\not{k}}\not{n} \tag{108a}$$

$$Y\text{-}Y\text{-}\Omega \text{ vertex} \qquad \frac{1}{\alpha}\gamma^i\frac{1-\gamma_3}{2}\frac{1}{4}(\Gamma_{ab}\not{n})_{\alpha\beta} \tag{108b}$$

while for the anomalous vertex we get from (107) the Feynman rule

$$Y\text{-}Y\text{-}\mathcal{L} \text{ anomalous vertex} \qquad -\frac{i}{\alpha}\frac{\hat{\not{k}} - \hat{\not{k}}'}{2}\gamma^3\frac{1}{4}(\Gamma_{cd}\not{n})_{\alpha\beta}. \tag{108c}$$

The anomaly can now be computed in the same way as in the preceding section, one only has to flip the chiralities. The first diagram in Fig. 1 with the insertion of the anomalous vertex (108c) and one external Ω_{jab} gives

$$\frac{i}{\alpha}\mathcal{A}^{cd}_{jab}(p) = \frac{i}{32}\,\mathrm{tr}\left(\Gamma^{cd}\not{n}\not{n}\Gamma_{ab}\not{n}\not{n}\right)\int\frac{d^Dk}{(2\pi)^D}\,\mathrm{tr}\left(\hat{\not{k}}\gamma^3\frac{1}{\not{k}}\gamma_j\frac{1-\gamma_3}{2}\frac{1}{\not{k}-\not{p}}\right). \tag{109}$$

The integral in (109) has already been calculated in the previous section (see (77)) while the trace of Γ-matrices, apart from terms which go to zero due to (54), can be calculated to give

$$\mathrm{tr}\left(\Gamma^{cd}\not{n}\not{n}\Gamma_{ab}\not{n}\not{n}\right) = -16\delta^c_{[a}\delta^d_{b]}. \tag{110}$$

The result for $\epsilon \to 0$ is

$$\mathcal{A}^{cd}_{jab} = -\frac{\alpha}{8\pi}\delta^c_{[a}\delta^d_{b]}(\eta_{mj}+\varepsilon_{mj})(ip^m). \tag{111}$$

Adding the external legs $\mathcal{L}_{cd}$ and $\Omega_j{}^{ab}$ and restoring the right-accented zweibeins we get for the $SO(8)$ Lorentz anomaly

$$\mathcal{A}'_L = -\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{\acute{g}}\,\mathrm{tr}\left(\mathcal{L}\acute{D}_-\Omega_+\right). \tag{112}$$

Here the traces are in the fundamental representation of the Lorentz group, $\mathrm{tr}(\mathcal{L}\Omega_j) \equiv \mathcal{L}_{ab}\Omega_j{}^{ba}$. Eq. (112) gives the anomaly under $SO(8)$ transformations. We postulate that the anomaly under $SO(10)$ transformations, in an eventual covariant quantization scheme, is still given by (112) where the constraints (54) and (98) are released. Also in this case the diagrams with two or more external Ω_i fields and the insertion of an anomalous vertex are zero for $\epsilon \to 0$.

Again, to render the anomaly diff-invariant we add a trivial cocycle to the effective action as in (84)

$$\Gamma_F = \Gamma'_F - \frac{\alpha}{16\pi}\int d^2\sigma\sqrt{\acute{g}}\,\mathrm{tr}\left(\acute{\Omega}_-\Omega_+\right), \tag{113}$$

so that

$$\mathcal{A}_L = -\frac{\alpha}{8\pi}\int d^2\sigma\varepsilon^{ij}\,\mathrm{tr}\left(\mathcal{L}\partial_i\Omega_j\right). \tag{114}$$

The direct computation of the Lorentz anomaly in this case requires to compute the Ω-Ω contribution to the effective action coming from the integration over the

fermions Y^α in (106). The computation is standard, all one has to use is again (110) and the result is completely analogous to (87):

$$\Gamma'_F = -\frac{\alpha}{16\pi}\int d^2\sigma\sqrt{\acute{g}}\,\mathrm{tr}\left(\acute{D}_-\Omega_+\frac{1}{\Box_{\acute{g}}}\acute{D}_-\Omega_+\right). \tag{115}$$

By adding the trivial cocycle as in (113) we get an expression analogous to (89),

$$\Gamma_F = \frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(\Omega_+\frac{1}{D_+}\frac{\varepsilon^{ij}\partial_i\Omega_j}{\sqrt{g}}\right) \tag{116}$$

which is now diff-invariant. Its Lorentz variation is

$$\begin{aligned}\delta_L\Gamma_F = &-\frac{\alpha}{8\pi}\int d^2\sigma\varepsilon^{ij}\,\mathrm{tr}\left(\mathcal{L}\partial_i\Omega_j\right) - \\ &-\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(D_-[\mathcal{L},\Omega_+]\frac{1}{\Box_g}D_-\Omega_+\right).\end{aligned} \tag{117}$$

Again, the first line is the anomaly (114) while the second line is non-local and gets cancelled by the diagram with three external Ω's, see Fig. 1.

For a first attempt on the derivation of Eq. (117) see [8]. Let us briefly discuss the appearance of additional Lorentz anomalies. Generally speaking they can arise from the terms in (56) where the connection Ω_i appears explicitly. In the term $-\frac{1}{2}\sqrt{g}g^{ij}D_iy^aD_jy_a$ the connection is non-chirally coupled, so no Lorentz anomaly can arise. For what concerns the mixed term $-2\sqrt{g}D_-y^aV_+{}^\alpha(\Gamma_a)_{\alpha\beta}y^\beta$, to preserve manifest $SO(8)$ invariance we have to impose the physical condition on the external field $V_j{}^\beta$, $\not{n}_{\alpha\beta}V_j{}^\beta = 0$. Then upon inserting the identity $\not{n}\not{m}+\not{m}\not{n} = 1$, this term becomes $-4\sqrt{g}\partial_-y^an_aV_+{}^\beta\not{m}_{\beta\alpha}y^\alpha$ such that the connection drops due to (54). The seventh term in (56) contains Ω_i explicitly but does not contribute to the Lorentz anomaly as we will see in the next section.

The terms which are quadratic in the y^α in (56) give rise to "trivial" anomalies and do therefore not constitute "anomalies". We evidenciate this fact for the nineth term. To preserve $SO(8)$ invariance we have to impose on T_{abc} the condition

$$n_aT^{abc} = 0 = m_aT^{abc}. \tag{118}$$

Then this term can be taken into account simply by defining

$$\tilde{\Omega}_{ia}{}^b \equiv \Omega_{ia}{}^b - \frac{1}{2}e_{-i}V_+{}^gT_{ga}{}^b, \tag{119}$$

that is:

$$\begin{aligned}\tilde{\Omega}_{+a}{}^{b} &= \Omega_{+a}{}^{b} - V_{+}{}^{g} T_{ga}{}^{b} \\ \tilde{\Omega}_{-a}{}^{b} &= \Omega_{-a}{}^{b}.\end{aligned} \tag{120}$$

This would produce instead of (112) the anomaly

$$\begin{aligned}\tilde{\mathcal{A}}_L &= -\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{\acute{g}}\,\mathrm{tr}\left(\mathcal{L}\acute{D}_{-}\tilde{\Omega}_{+}\right) \\ &= \mathcal{A}'_L - \frac{\alpha}{8\pi}\int d^2\sigma\sqrt{\acute{g}}\,\acute{\partial}_{-}\mathcal{L}_{ab}V_{+}{}^{c}T_{c}{}^{ba} \\ &= \mathcal{A}'_L + \delta_L\left(-\frac{\alpha}{8\pi}\int d^2\sigma\sqrt{\acute{g}}\,\acute{\Omega}_{-ab}V_{+}{}^{c}T_{c}{}^{ba}\right)\end{aligned} \tag{121}$$

and therefore $\tilde{\mathcal{A}}_L$ and $\mathcal{A}_L$ represent the same cohomology class.

The Lorentz anomaly can be cancelled if we subject the two-superform B to the anomalous Lorentz transformation

$$\delta_L B = \frac{\alpha}{8\pi}\,\mathrm{tr}(\mathcal{L}d\Omega) \tag{122}$$

which, together with (95), defines the gauge and Lorentz invariant curvature

$$H = dB + \frac{\alpha}{8\pi}(\omega_{3YM} - \omega_{3L}) \tag{123}$$

with the associated Bianchi identity in superspace

$$dH = \frac{\alpha}{8\pi}\left(\mathrm{tr}\,F^2 - \mathrm{tr}\,R^2\right). \tag{124}$$

Notice that both traces in (124) are in the fundamental representations of $SO(32)$ and $SO(10)$ respectively and, according to the Green-Schwarz anomaly cancellation mechanism, this is then also precisely the relation which assures the absence of gauge and Lorentz anomalies in $N = 1$, $D = 10$ Supergravity-Super-Yang-Mills theory.

In the next section we will show that (123), (124) are actually sufficient and necessary to cancel also the Lorentz κ-anomaly in our sigma model.

7 The Lorentz-type κ-anomaly

At this point an important difference between the gauge sector and the gravitational sector shows up. The gauge-type κ-anomaly could be calculated by simply varying (87) while the Lorentz-type κ-anomaly can not be computed by varying simply (115). This can easily be seen by observing that in (115) with respect to (87) the chiralities are flipped. For the κ-transformations of the induced connections we have

$$\delta_\kappa A_i = \mathcal{D}_i C + F_i$$

$$\delta_\kappa \Omega_{ia}{}^b = D_i L_a{}^b + R_{ia}{}^b$$

where in both cases, see (38), $F_- = 0$ and $R_{-a}{}^b = 0$, while F_+ and $R_{+a}{}^b$ are different from zero. Therefore the variation of Γ_F gives, unlike as in the Yang-Mills case, apart from a local contribution, non-local contributions proportional to R_+; moreover Γ_F depends non-locally on $e_+{}^j$ and the κ-variation of $e_+{}^j$ induces additional non-local terms. It can also be seen that the local terms in $\delta_\kappa \Gamma_F$ do not satisfy the Wess-Zumino consistency condition, see the next section.

The key observation for the resolution of this puzzle is that, as can be seen from (45), κ-transformations mix the fermions y^α with the bosons y^a. The Lorentz-type κ-anomaly stems from the explicit coupling of the induced Lorentz-connection Ω_i to the quantum fields (y^a, y^α). While the y^a do not contribute to the Lorentz-anomaly, as we mentioned already, they are expected to contribute to the Lorentz-type κ-anomaly because of their explicit coupling to the Ω_i in the term $-\frac{1}{2}\sqrt{g} g^{ij} D_i y^a D_j y_a$. Their contribution is actually essential to saturate the coupled cohomology problem (1). The analogy with the supersymmetric partner of an ABBJ anomaly in a $d = 2$ Super-Yang-Mills theory has already been discussed in the introduction.

Since massless scalars in two dimensions, as are the y^a, are always plagued by infrared divergences we introduce an infrared mass regulator m and take the relevant boson action to be

$$I_B = -\frac{1}{2} \int d^2\sigma \sqrt{g} \left(g^{ij} D_i y^a D_j y_a - m^2 y^a y_a \right). \tag{125}$$

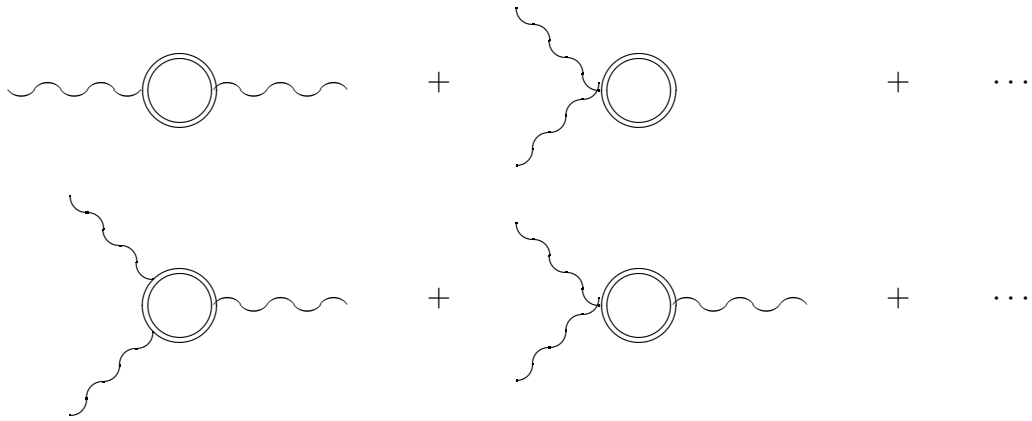

Figure 3: Bosonic graphs contributing to the one-loop effective action.

Remember that $D_i y^a = \partial_i y^a + y^b \Omega_{ib}{}^a$. The Feynman rules for $g^{ij} = \eta^{ij}$ are

$$y^a \text{ propagator} \qquad -\frac{i\alpha}{k^2 - m^2}\eta_{ab} \tag{126a}$$

$$\Omega\text{-}y\text{-}y \text{ vertex} \qquad \frac{1}{\alpha}(k + k')i\delta^c_{[a}\delta^d_{b]} \tag{126b}$$

$$\Omega\text{-}\Omega\text{-}y\text{-}y \text{ vertex} \qquad -\frac{2i}{\alpha}\eta^{ij}\eta_{bd}\eta_{fa}\eta_{gc}. \tag{126c}$$

The last vertex has to be saturated with the external legs $\Omega_i{}^{ab}\Omega_j{}^{cd}$ while f and g indicate the internal boson lines.

We compute the contribution of (125) to the effective action which is quadratic in the Ω_i. We have a self-energy type diagram and a tadpole diagram (the first two pictures in Fig. 3). Since each of the two diagrams is individually ultraviolet divergent we introduce also here a dimensional regularization with $D = 2 + \epsilon$ and a scale μ to compute them. Adding up the two diagrams we get in momentum space for generic m and ϵ

$$\frac{1}{\alpha}\Gamma_{iab;j}{}^{cd}(p) = \delta^d_{[a}\delta^c_{b]}\left(\eta_{ij} - \frac{p_i p_j}{p^2}\right)B(p^2) \tag{127}$$

where

$$B(p^2) = \frac{\Gamma(-\epsilon/2)}{(4\pi)^{D/2}}\left(\frac{m}{\mu}\right)^{\epsilon}\int_0^1 dx\left[\left(1 - x(1-x)\frac{p^2}{m^2}\right)^{\epsilon/2} - 1\right]. \tag{128}$$

The result (127) is transverse as is required by the target-space Lorentz invariance of (125). If we take m fixed and send $\epsilon \to 0$ the function B admits a finite limit

meaning that the ultraviolet divergences which are present in both diagrams (Fig. 3) cancel each other. Explicitly we get

$$\begin{aligned} B\Big|_{\epsilon=0} &= -\frac{1}{4\pi}\int_0^1 dx \ln\left(1 - x(1-x)\frac{p^2}{m^2}\right) \\ &= -\frac{1}{4\pi}\int_0^1 dx \ln\left(-\frac{m^2}{p^2} + x(1-x)\right) - \frac{1}{4\pi}\ln\left(-\frac{p^2}{m^2}\right). \end{aligned} \tag{129}$$

However, this result does not admit a finite limit for $m \to 0$ which signals the presence of an infrared divergence as anticipated above. For $m \to 0$ the divergence can be directly read off from (129)

$$\lim_{m\to 0} B\Big|_{\epsilon=0} \to \frac{1}{4\pi}\left(2 - \ln\left(-\frac{p^2}{m^2}\right)\right). \tag{130}$$

Alternatively in (128) we can first send $m \to 0$ and then regularize the infrared divergence with the dimensional regularization which is already present

$$B\Big|_{m=0} = \frac{1}{4\pi}\left(-\frac{p^2}{4\pi\mu^2}\right)^{\epsilon/2}\Gamma(-\epsilon/2)\int_0^1 dx (x(1-x))^{\epsilon/2}.$$

Sending now $\epsilon \to 0$ the infrared divergence shows up as a simple pole in ϵ

$$\lim_{\epsilon\to 0} B\Big|_{m=0} \to \frac{1}{4\pi}\left(2 - \ln\left(-\frac{p^2}{4\pi\mu^2}\right) - \frac{2}{\epsilon} - \gamma\right) \tag{131}$$

where γ is Euler's constant.

To our knowledge infrared divergences of this type have not yet been discussed in string theory and at present we have no proof for their cancellation. Below we will argue that these divergences are actually only perturbative effects. Comparing (130) with (131) we can separate out the infrared divergence and determine the finite part of B to be

$$B_f = \frac{1}{2\pi}. \tag{132}$$

In writing (132) we omitted the term $\ln(-p^2)$ and the other (finite and divergent) parts which we interpret as infrared effects, for the discussion see below. A similar criterium for the separation of infrared divergences has been adopted in [14] to prove the absence of a level shift in the WZWN model at two loops. In our

case (132) is actually the unique choice which leads to a Wess-Zumino consistent anomaly as we will see in the next section. With (132) we get for (127)

$$\Gamma_{iab;j}{}^{cd} = \frac{\alpha}{2\pi}\delta^d_{[a}\delta^c_{b]}\left(\eta_{ij} - \frac{p_i p_j}{p^2}\right).$$

Upon adding the external legs we obtain for the boson contribution to the effective action

$$\Gamma_B = \frac{\alpha}{4\pi}\int d^2\sigma\,\mathrm{tr}\left(\Omega_i\left(\eta^{ij} - \frac{\partial^i\partial^j}{\Box}\right)\Omega_j\right)$$

and by restoring the worldsheet metric we get

$$\Gamma_B = \frac{\alpha}{16\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left[\left(D_-\Omega_+ - D_+\Omega_-\right)\frac{1}{\Box_g}\left(D_-\Omega_+ - D_+\Omega_-\right)\right]. \tag{133}$$

The total effective action can now be computed from (116) and (133) to be

$$\begin{aligned}\Gamma &= \Gamma_F + \Gamma_B \\ &= \frac{\alpha}{16\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(D_+\Omega_-\frac{1}{\Box_g}(D_+\Omega_- - D_-\Omega_+)\right) \\ &= \frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\,\mathrm{tr}\left(\Omega_-\frac{1}{D_-}\frac{\varepsilon^{ij}\partial_i\Omega_j}{\sqrt{g}}\right)\end{aligned} \tag{134}$$

which is now, apart from a sign difference due to the opposite chirality of the heterotic fermions and the y^α, formally identical to the effective action gotten from the integration over the heterotic fermions, see (89). In particular (134) does not depend on $e_+{}^i$, but only on the κ-invariant fields $e_-{}^i$ and $\sqrt{g}$. Therefore, when computing the κ-variation of (134) it is not necessary to vary the worldsheet metric, but we can limit ourselves to vary the induced connection Ω_i. To understand better the non-local contributions of this variation we vary Γ_F and Γ_B separately

$$\begin{aligned}\delta_\kappa\Gamma_F &= -\frac{\alpha}{8\pi}\int d^2\sigma\varepsilon^{ij}\,\mathrm{tr}\left(L\partial_i\Omega_j - R_i\Omega_j\right) \\ &\quad - \frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\left[\mathrm{tr}\left(D_-\left[L,\Omega_+\right]\frac{1}{\Box_g}D_-\Omega_+\right) + \mathrm{tr}\left(D_-R_+\frac{1}{\Box_g}D_-\Omega_+\right)\right]\end{aligned} \tag{135a}$$

$$\begin{aligned}\delta_\kappa\Gamma_B &= -\frac{\alpha}{8\pi}\int d^2\sigma\,2\varepsilon^{ij}\,\mathrm{tr}\left(R_i\Omega_j\right) \\ &\quad + \frac{\alpha}{8\pi}\int d^2\sigma\sqrt{g}\left[\mathrm{tr}\left(\left(D_-\left[L,\Omega_+\right] - D_+\left[L,\Omega_-\right]\right)\frac{1}{\Box_g}\left(D_-\Omega_+ - D_+\Omega_-\right)\right)\right.\end{aligned}$$

$$+\,\mathrm{tr}\left(D_-R_+\frac{1}{\Box_g}D_-\Omega_+\right)\Bigg]\,. \tag{135b}$$

Now let us discuss the non-local terms in (135a) and (135b); first we notice that the non-local terms proportional to R_+ cancel between (135a) and (135b). The term proportional to $[L,\Omega_+]$ in (135a) is cancelled by the κ-variation of the Ω-Ω-Ω contribution to the effective action gotten from the integration over the fermionic y^α since this term is due to the (field-dependent) Lorentz transformation contained in the κ-transformation, and as we saw in the preceding section (see formula (117)), the Ω-Ω-Ω contribution does not affect the Lorentz anomaly. This is completely analogous to the case of the heterotic fermions. The non-local contributions in (135b) which are proportional to $[L,\Omega_\pm]$ are cancelled by the (Lorentz part of) the variation of the Ω-Ω-Ω contribution to the effective action gotten by the integration over the bosonic y^a, simply because the y^a do not contribute to the Lorentz anomaly. Adding up the remaining contributions, which are all local, we get for the κ-anomaly

$$\begin{aligned}\mathcal{A}_L^\kappa &= -\frac{\alpha}{8\pi}\int d^2\sigma\varepsilon^{ij}\,\mathrm{tr}\,(L\partial_i\Omega_j+R_i\Omega_j)\\ &= \frac{\alpha}{16\pi}\int d^2\sigma\varepsilon^{ij}V_i{}^AV_j{}^B\Delta^\gamma(\omega_{3L})_{\gamma BA}\end{aligned} \tag{136}$$

where we used the super Lorentz-Chern-Simons form defined in (8).

Clearly the result can also be obtained by varying directly (134) and keeping only the local terms. The anomaly in (136) can be eliminated in the same way as the Yang-Mills type κ-anomaly in section V. The anomaly (136) can be cancelled if we modify once more Eq. (94) defining a new three-form field strength H according to

$$H = dB + \frac{\alpha}{8\pi}\left(\omega_{3YM}-\omega_{3L}\right) \tag{137}$$

and impose on H defined in (137) the constraints

$$\begin{aligned}H_{\alpha\beta\gamma} &= H_{ab\alpha} = 0\\ H_{a\alpha\beta} &= 2(\Gamma_a)_{\alpha\beta}.\end{aligned} \tag{138}$$

Notice that (137) coincides with the definition (123), i.e. precisely the relation which ensures also the cancellation of gauge and Lorentz anomalies.

We will comment on possible additional "true" one-loop κ-anomalies in the next section. Here we would like to point out that at one-loop the effective action can produce trivial κ-anomalies which have to be eliminated by performing suitable local subtractions on the classical action (10). We will illustrate this fact in the following example.

In fact, additional contributions to the one-loop effective actions can be computed by observing that the seventh and nineth term in (56) correspond formally to a shift of the connection $\Omega_{ia}{}^{b}$ in the sense that they can be absorbed in the second and first term respectively by defining formally a new Lorentz connection as

$$\tilde{\Omega}_{ia}{}^{b} \equiv \Omega_{ia}{}^{b} - \frac{1}{2} e_{-i} V_{+}{}^{g} T_{ga}{}^{b}. \tag{139}$$

Therefore the seventh and nineth term in (56) can be taken into account by replacing in the fermionic contribution (115) and in the bosonic contribution (133) Ω_i with $\tilde{\Omega}_i$ to get respectively $\tilde{\Gamma}'_F$ and $\tilde{\Gamma}_B$. Summing up we obtain

$$\tilde{\Gamma}'_F + \tilde{\Gamma}_B = \Gamma + \frac{\alpha}{16\pi} \int d^2\sigma \sqrt{\acute{g}} \left[\left(\Omega_{+a}{}^{b} - 2V_{+}{}^{g} T_{ga}{}^{b} \right) \acute{\Omega}_{-b}{}^{a} + \frac{1}{2} \delta^{i}_{-} e_{-i} V_{+}{}^{g} T_{ga}{}^{b} V_{+}{}^{h} T_{hb}{}^{a} \right], \tag{140}$$

and the last three terms in this formula are not κ-invariant, but local. Therefore the seventh and nineth term give rise to a trivial κ-anomaly which has to be eliminated by redefining the classical action according to

$$I \to I - \frac{\alpha}{16\pi} \int d^2\sigma \sqrt{\acute{g}} \left[\left(\Omega_{+a}{}^{b} - 2V_{+}{}^{g} T_{ga}{}^{b} \right) \acute{\Omega}_{-b}{}^{a} + \frac{1}{2} \delta^{i}_{-} e_{-i} V_{+}{}^{g} T_{ga}{}^{b} V_{+}{}^{h} T_{hb}{}^{a} \right]. \tag{141}$$

Notice that the first two cocycles in (141) are precisely those which had to be subtracted in the previous section to get a diff-invariant Lorentz *anomaly*, see (113) and (121); the last cocycle in (141) is Lorentz invariant and is needed to cancel a diff-anomaly from the *effective action*.

Let us now briefly comment on the infrared divergence encountered above. The divergence is due to the presence of scalar massless bosons, the y^a which in two dimensions are known to be plagued by infrared divergences. We argue that in the case at hand these divergences are actually perturbative effects by reasoning as follows. In our case, in fact, the fields y^a are "essentially" massive,

in the sense that there are terms in the action (56) which are quadratic in the y^a

$$\frac{1}{2}\int d^2\sigma\sqrt{g}\, y^a \mathcal{M}^2_{ab}(\sigma) y^b \tag{142}$$

where $\mathcal{M}_{ab}$ is a function of the external fields. Let us assume that there exists a configuration of the external fields such that $\mathcal{M}^2_{ab}(\sigma)$ becomes a constant matrix, i.e. independent of σ, and let us also assume, for the sake of simplicity, that this matrix is proportional to the identity

$$\mathcal{M}^2_{ab}(\sigma) = \mathcal{M}^2 \delta_{ab}. \tag{143}$$

Then, for this configuration, (142) produces a mass term for the scalars, with mass $\mathcal{M}$. Then no infrared regularization is required and formula (128) becomes

$$\begin{aligned} B_{\epsilon=0} &= -\frac{1}{4\pi}\int_0^1 dx \ln\left(-\frac{\mathcal{M}^2}{p^2} + x(1-x)\right) - \frac{1}{4\pi}\ln\left(-\frac{p^2}{\mathcal{M}^2}\right) \\ &= \frac{1}{2\pi} - \frac{1}{4\pi}\int_0^1 dx \ln\left(\frac{1}{x(1-x)} - \frac{p^2}{\mathcal{M}^2}\right). \end{aligned} \tag{144}$$

The integral in (144) is now convergent, but it is *non-analytic* in the "external fields" $\mathcal{M}$. The perturbative approach we adapted to compute the Lorentz-type κ-anomaly was based on a power series expansion in terms of polynomials in the external fields, but clearly (144) cannot be expanded, around $\mathcal{M} = 0$, in polynomials of $\mathcal{M}$. If one can generalize this argument for a generic configuration of the external fields and we guess that this is possible, then one can conclude that an additive part of the effective action is non-analytic in the external fields and the infrared divergences we encountered are just signals of this non-analyticity. The κ-invariance of the non-analytic contribution to the effective action seems rather difficult to control, we guess that it is actually invariant due to the fact that anomalies should always be local, and hence analytic.

As a last remark of this section we would like to stress that extracting as "analytic" part from (144) the constant $1/2\pi$ turns out to be actually the correct choice because the anomaly computed with this constant, and only with this constant, turns out a) to be local and b) to satisfy the Wess-Zumino consistency condition. In fact, for a different constant the non local-terms proportional to R_+ would not cancel between (135a) and (135b).

A cohomogical analysis of the computed κ-anomalies and a brief discussion of the resulting SUGRA-SYM theory follows in the next section.

8 Wess-Zumino consistency condition and SUGRA-SYM theory

As anticipated in the introduction the computation of one-loop κ-anomalies permits, imposing their cancellation, to derive the order-α corrections to the classical constraints on the superfields of the background theory. As has been shown in [2] the Wess-Zumino consistency condition which has to be satisfied by the κ-anomalies ensures the solvability of the Bianchi identities with these new constraints.

In this section we want to describe the main features of this method to derive in particular the consistent order-α corrections to the pure $N = 1$, $D = 10$ SUGRA-SYM theory and apply it to the anomalies we have computed.

The total anomaly computed in the previous sections can be written as

$$\mathcal{A}_\kappa = -\frac{\alpha}{16\pi}\int d^2\sigma \varepsilon^{ij} V_i{}^A V_j{}^B \Delta^\gamma G_{\gamma BA} \tag{145}$$

where $G_{\gamma BA}$ are the components of the three-superform

$$G = \frac{1}{3!} E^A E^B E^C G_{CBA} \equiv \omega_{3YM} - \omega_{3L} \tag{146}$$

satisfying

$$dG = \mathrm{tr}\, F^2 - \mathrm{tr}\, R^2. \tag{147}$$

By taking for the BRS transformations of the ghosts $\kappa_{+\alpha}$ (the ghosts $\kappa_{+\alpha}$ commute between themselves, $\kappa_{+\alpha}\kappa_{+\beta} = \kappa_{+\beta}\kappa_{+\alpha}$)

$$\delta_\kappa \kappa_{+\alpha} = \kappa_{+\beta}\kappa_{+\gamma}\left(\not{V}_-^{\beta\varepsilon}\Omega_{\varepsilon\alpha}{}^\gamma + \delta_\alpha^\beta(\not{V}\lambda)^\gamma - \not{V}_-^{\beta\gamma}\lambda_\alpha + 4\delta_\alpha^\beta V_-{}^\gamma - (\Gamma_g)^{\beta\gamma}(\Gamma^g)_{\alpha\varepsilon}V_-{}^\varepsilon\right), \tag{148}$$

we can construct an on-shell nihilpotent BRS operator Ω_κ, satisfying $\Omega_\kappa^2 = 0$ (on shell). Then the anomaly is characterized as a (non-trivial) cocycle of Ω_κ

satisfying the BRS consistency condition

$$\Omega_\kappa \mathcal{A}_\kappa = 0. \tag{149}$$

By rewriting (145) as

$$\mathcal{A}_\kappa = -\frac{\alpha}{16\pi}\int d^2\sigma \varepsilon^{ij}\partial_i Z^M \partial_j Z^N \delta_\kappa Z^L G_{LNM},$$

we can compute (149), which turns out to be, modulo terms proportional to the equations of motion,

$$\begin{aligned}\Omega_\kappa \mathcal{A}_\kappa &= -\frac{\alpha}{8\pi}\int d^2\sigma \varepsilon^{ij}\partial_i Z^M \partial_j Z^N \delta_\kappa Z^L \delta_\kappa Z^P \partial_{[P} G_{LNM)} \\ &= -\frac{\alpha}{32\pi}\int d^2\sigma \varepsilon^{ij} V_i{}^A V_j{}^B \Delta^\alpha \Delta^\beta (dG)_{\beta\alpha BA} = 0. \end{aligned} \tag{150}$$

Due to the constraints (6c), (6d) with (147) the condition (150) reduces to

$$V_-{}^c V_+{}^d \Delta^\alpha \Delta^\beta \left(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2\right)_{\alpha\beta cd} = 0$$

which, under the constraints (6), (7), becomes

$$\Delta^\alpha \not{V}_{-\alpha\gamma}\left[\mathrm{tr}(\chi^\gamma\chi^\delta) - \mathrm{tr}(T^\gamma T^\delta)\right]\not{V}_{+\delta\beta}\Delta^\beta = 0 \tag{151}$$

where we wrote $\mathrm{tr}(T^\gamma T^\delta) \equiv T_{ab}{}^\gamma T^{\delta ba}$. On-shell (151) is identically satisfied due to Eq. (19) and (37d) so that under the constraints (6), (7) our anomaly satisfies the consistency condition identically.

In [2] it has been shown that for a generic G satisfying (150) the Bianchi identities can be consistently solved with the constraints (138) and the definition $H = dB + \frac{\alpha}{8\pi}G$. Then the Bianchi identities (124) can be consistently solved with the constraints (138) while the constraints (6a), (6c) remain unchanged. The check of the consistency of the Bianchi identities is straightforward, here we report the order-α corrected relations between the various superfields. Notice that it is not consistent to keep α^2-corrections in that for getting the complete α^2-corrections one had to compute two-loop anomalies together with other arrangements, see the discussion in the concluding section. We get

$$T_{a\alpha}{}^\beta = \frac{1}{4}(\Gamma_{bc})_\alpha{}^\beta T_a{}^{bc} - \frac{\alpha}{16\pi}(\Gamma_a)_{\alpha\varepsilon}\left(\mathrm{tr}(\chi^\varepsilon\chi^\beta) - \mathrm{tr}(T^\varepsilon T^\beta)\right) \tag{152a}$$

$$D_\alpha T_{abc} = (\Gamma_{[a})_{\alpha\beta}\left(-6T_{bc]}{}^{\beta} - \frac{3\alpha}{8\pi}\left(\mathrm{tr}(F_{bc]}\chi^\beta) - \mathrm{tr}(R_{bc]}T^\beta)\right)\right) \tag{152b}$$

$$\begin{aligned} D_\alpha\lambda_\beta &= -(\Gamma_g)_{\alpha\beta}D^g\phi + \lambda_\alpha\lambda_\beta \\ &\quad + \frac{1}{12}(\Gamma_{abc})_{\alpha\beta}\left[T^{abc} + \frac{\alpha}{64\pi}(\Gamma^{abc})_{\gamma\delta}\left(\mathrm{tr}(\chi^\gamma\chi^\delta) - \mathrm{tr}(T^\gamma T^\delta)\right)\right] \end{aligned} \tag{152c}$$

$$R_{\alpha\beta ab} = -\frac{\alpha}{8\pi}(\Gamma_{[a})_{\alpha\varepsilon}\left(\mathrm{tr}(\chi^\varepsilon\chi^\varphi) - \mathrm{tr}(T^\varepsilon T^\varphi)\right)(\Gamma_{b]})_{\varphi\beta} \tag{152d}$$

$$R_{a\alpha bc} = 2(\Gamma_a)_{\alpha\beta}T_{bc}{}^{\beta} + \frac{3\alpha}{16\pi}(\Gamma_{[a})_{\alpha\beta}\left[\mathrm{tr}(F_{bc]}\chi^\beta) - \mathrm{tr}(R_{bc]}T^\beta)\right]. \tag{152e}$$

In particular we have again

$$H_{abc} = T_{abc}. \tag{153}$$

With respect to the zeroth order constraints the principal feature is the appearance of a non-vanishing $R_{\alpha\beta ab}$, which acquires now a 120 irreducible representation (irrep) of $SO(10)$, as is expected on general grounds for non-minimal supergravity theories, see [21, 15, 18]. Notice that now $(\mathrm{tr}\, R^2)_{\alpha\beta\gamma\delta}$ and $(\mathrm{tr}\, R^2)_{\alpha\beta\gamma a}$ are no longer zero, but of order α and hence the Wess-Zumino condition (149) is no longer satisfied identically: it is satisfied only at first order in α according to our one-loop computation. We stress again that to take α^2-corrections into account one had to go to two-loops.

Let us now discuss the presence of possible additional "true" anomalies at first order in α, i.e. at one loop. For this purpose it is convenient to recall that the total κ-anomaly $\mathcal{A}_\kappa^T$ and the gauge and Lorentz anomalies $\mathcal{A}_G$ and $\mathcal{A}_L$ satisfy on general grounds the following coupled cohomology problem

$$\Omega_\kappa\mathcal{A}_\kappa^T = 0, \qquad \Omega_L\mathcal{A}_G + \Omega_G\mathcal{A}_L = 0 \tag{154a}$$

$$\Omega_G\mathcal{A}_G = 0, \qquad \Omega_L\mathcal{A}_\kappa^T + \Omega_\kappa\mathcal{A}_L = 0 \tag{154b}$$

$$\Omega_L\mathcal{A}_L = 0, \qquad \Omega_G\mathcal{A}_\kappa^T + \Omega_\kappa\mathcal{A}_G = 0 \tag{154c}$$

where Ω_κ, Ω_G, Ω_L are the BRS operators associated to κ, gauge and Lorentz transformations respectively. If we take for $\mathcal{A}_\kappa^T$ the anomaly $\mathcal{A}_\kappa$ we have found, see Eq. (145), and for $\mathcal{A}_G$ and $\mathcal{A}_L$ (85) and (114) respectively it is not difficult to show that all the equations in (154) are indeed satisfied, the first equation in (154a) is nothing else than (149). Now, the gauge and Lorentz anomalies (85)

and (114) are expected to be exact, i.e. not to get higher-loop corrections and clearly they are one-loop exact, but it is not obvious at all that Eq. (145) presents the complete one-loop κ-anomaly. We can in general write

$$\mathcal{A}_\kappa^T = \mathcal{A}_\kappa + X_\kappa \tag{155}$$

where X_κ is a possible missing anomaly. Then (155) has to satisfy again (154) and, using the fact that $\mathcal{A}_\kappa$ satisfies it already, we get the conditions:

$$\begin{aligned} \Omega_\kappa X_\kappa &= 0 \\ \Omega_G X_\kappa &= 0 \\ \Omega_L X_\kappa &= 0 \end{aligned} \tag{156}$$

which means that the missing anomaly X_κ has to be gauge and Lorentz invariant and that it has to satisfy the κ-consistency condition independently from $\mathcal{A}_\kappa$. Possible solutions to (156) can be constructed as follows. We write

$$X_\kappa = \frac{1}{2}\int d^2\sigma \varepsilon^{ij} V_i{}^A V_j{}^B \Delta^\gamma X_{\gamma BA} \tag{157}$$

where the two V's have to be there for dimensional reasons and we take X_{CBA} to be the components of a three-superform

$$X = \frac{1}{3!} E^A E^B E^C X_{CBA}$$

which has to be *gauge and Lorentz invariant*. The κ-consistency condition on X_κ becomes then

$$\int d^2\sigma \varepsilon^{ij} V_i{}^A V_j{}^B \Delta^\gamma \Delta^\delta (dX)_{\delta\gamma BA} = 0, \tag{158}$$

which is equivalent to

$$(dX)_{\alpha\beta\gamma\delta} = 0 \tag{159a}$$

$$(dX)_{\alpha\beta\gamma a} = 0 \tag{159b}$$

$$\Delta^\beta V_-{}^a (dX)_{\alpha\beta ab} V_+{}^b \Delta^\alpha = 0. \tag{159c}$$

Once the first two equations are satisfied the third one can be shown to be equivalent to

$$(dX)_{\alpha\beta ab} = (\Gamma_{[a})_{\alpha\varphi} H^{\varphi\delta} (\Gamma_{b]})_{\delta\beta} \tag{159d}$$

for some antisymmetric superfield $H^{\varphi\delta}$ belonging to the 120 dimensional irreducible representation of $SO(10)$ (see [2] and the previous section). A class of solutions of Eqs. (159) can be determined as follows. Let us consider a gauge-invariant (and Lorentz covariant) superfield $Y^{abcd}(Z)$ which is antisymmetric in all its indices and belongs therefore to the 210 irrep of $SO(10)$. Let us assume, moreover that the combination $D_\alpha Y_{abcd} + 2\lambda_\alpha Y_{abcd}$ does not contain the highest 1440-dimensional irrep of $SO(10)$, i.e.

$$(D_\alpha Y_{abcd} + 2\lambda_\alpha Y_{abcd})^{1440} = 0. \tag{160}$$

Then we can construct an X satisfying (158) in the following way:

$$\begin{aligned} X_{\alpha\beta\gamma} &= 0 \\ X_{a\alpha\beta} &= (\Gamma_{abcde})_{\alpha\beta} Y^{bcde}. \end{aligned} \tag{161}$$

At this point it is not difficult to show that Eqs. (159) determine *consistently and uniquely* $X_{ab\alpha}$ and X_{abc}.

To conclude: each (gauge-invariant and Lorentz covariant) 210 irrep satisfying (160) specifies uniquely a cocycle of the operator Ω_κ, and hence a possible anomaly. If X can not be written as the superdifferential of a two-superform $\tilde{B}$, $X \neq d\tilde{B}$, then X corresponds to a non trivial cocycle, i.e. to a true anomaly (otherwise it can be eliminated by redefining the Wess-Zumino two-form B). In this last case the anomaly (157) can be eliminated by imposing on H, still defined in (137), the constraints

$$\begin{aligned} H_{\alpha\beta\gamma} &= 0 \\ H_{a\alpha\beta} &= 2\Gamma_{a\alpha\beta} + X_{a\alpha\beta} \\ H_{ab\alpha} &= X_{ab\alpha}. \end{aligned} \tag{162}$$

In particular the relation between H_{abc} and T_{abc} becomes now

$$H_{abc} = T_{abc} + X_{abc} \tag{163}$$

instead of (153). Eq. (158) assures again that the Bianchi identities can be consistently solved, in particular the field $H^{\varphi\delta}$ modifies the relations given in (152) by additional terms on the r.h.s., proportional to $H^{\varphi\delta}$.

Are such additional κ-anomalies really present at one-loop in our sigma-model? The results of Ref. [15] could suggest that such an additional κ-anomaly should show up. That paper deals with the solution of the Bianchi identity

$$dH = \frac{\alpha}{8\pi}\left(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2\right) \tag{164}$$

at second order in α^2 (actually this paper gives a complete all order solution of this Bianchi identity, found previously in [22] with a different but equivalent set of constraints for the superfields). It turns out that an all order solution can be obtained if one modifies the constraints on H precisely according to (162) where $X_{a\alpha\beta}$ and $X_{ab\alpha}$ are of *first* order in α, and, in particular, at first order in α the authors of [15] got for the 210 irrep Y_{abcd} appearing in (161)

$$Y_{abcd} = c\alpha\left(R_{[abcd]} + T_{[ab}{}^{\alpha}(\Gamma_{cd]})_{\alpha}{}^{\beta}\lambda_{\beta}\right), \tag{165}$$

where c is a constant. It can easily be verified that the Y_{abcd} given in this formula verifies indeed (160) up to order α and therefore the three-superform X constructed from (165) defines a cocycle of Ω_κ at first order in α. Then one could think that in the Green-Schwarz sigma model there should actually be an additional one-loop κ-anomaly, parametrized by (165). However, as will be shown elsewhere [19], the anomaly defined uniquely through Eqs. (165), (161) and (159) is a *trivial* anomaly at first order in α. Correspondingly the solution of the H-Bianchi identity found in [15] can be shown to be equivalent, at first order in α, to the solution found by us in Eqs. (152) and (153) in the sense that one solution can be mapped to the other through a redefinition of the fields of the SUGRA-SYM theory [19].

Therefore we expect that no non-trivial X_κ satisfying (156) should appear at one-loop in our sigma model; correspondingly the complete order-α corrections to the pure SUGRA-SYM theory are given in Eqs. (152), (153) which show a complete symmetry between the Yang-Mills and supergravity sectors. The equations of motion can be derived in a straightforward way from those relations using standard superspace techniques [22].

Clearly non-trivial anomalies satisfying (156) have to appear at order α^2,

i.e. at two loops in the sigma model, because the Bianchi identities with the parametrizations (152) are satisfied only at first order in α.

9 Conclusions

In this paper we established firmly the presence of the super Lorentz Chern-Simons form in the Green-Schwarz heterotic string sigma model in a SUGRA-SYM background. It has to be present in the definition of the field strength associated to the two-form superpotential B in order to cancel a one-loop κ-anomaly in the sigma model and also in order to cancel the one-loop Lorentz anomaly. The absence of κ-anomalies is a consistency requirement in the sigma model because κ-invariance ensures the decoupling of the eight unphysical degrees of freedom of the sixteen fermionic ϑ^μ variables. To guarantee this decoupling also at the quantum level we have to require the absence of κ-anomalies.

The relations (123) and (124), which entail the absence of gauge, Lorentz and κ-anomalies at one-loop, reduce in ordinary ten-dimensional space-time precisely to the relations which ensure the absence of the space-time gauge and Lorentz anomalies in $N = 1$, $D = 10$ SUGRA-SYM according to the Green-Schwarz mechanism [23].

The results of Refs. [2] imply moreover that the Bianchi identity (124) can be consistently solved with the constraints (138) and this implies in turn that one gets equations of motion in superspace which define a supersymmetric theory.

As we observed in section VIII no other true anomalies are expected to appear at one loop, but at two loops anomalies of the X-type, Eq. (156) have to show up for the reasons explained in that section. The computation of these two-loop anomalies would require the following technical arrangements.

a) The normal coordinate expansion, performed in section IV, contains a chiral gauge rotation of the heterotic fermions, with parameter Λ given in (39b) and an (implicit) chiral Lorentz rotation for the fermions y^α with parameter Σ. These rotations, as shown in [3] do not leave the functional fermion integral invariant,

and therefore, when making two-loop computations, the two corresponding Wess-Zumino actions have to be taken into account.

b) *All* trivial one-loop κ-cocycles have to be subtracted from the classical action (10) and normal coordinate expanded up to second order in y^A. The trivial cocycles we found entail a subtraction ΔI which is given by

$$\Delta I = -\frac{\alpha}{16\pi}\int d^2\sigma \left(\sqrt{\grave{g}}\,\mathrm{tr}(A_-\grave{A}_+) + \sqrt{\acute{g}}\,\mathrm{tr}(\acute{\Omega}_-\Omega_+)\right) - \\ - \frac{\alpha}{16\pi}\int d^2\sigma\sqrt{\acute{g}}V_+{}^g T_{ga}{}^b\left(-2\acute{\Omega}_{-b}{}^a + \frac{1}{2}\delta^i_- e_{-i}V_+{}^h T_{hb}{}^a\right). \qquad (166)$$

Notice, however, that this does not necessarily correspond to the whole one-loop subtraction one should make in that we did not perform a complete one-loop analysis of the effective action.

c) The action should be normal coordinate expanded up to the fourth-order in y^A. The order-α^2 anomaly gets contributions at one loop from the y^2 terms when one inserts the new constraints/parametrizations (152); in particular the three-form dB appearing in the normal coordinate expanded action at first-order in y^A has to be substituted with $H - \frac{\alpha}{8\pi}(\omega_{3YM} - \omega_{3L})$. The order-$\alpha^2$ anomaly gets contributions also at two loops from the y^3 and y^4 terms in which one has to insert the old classical constraints.

It seems to us, however, that this program, even if conceptually not too complicated, is technically rather involved.

It may also be that to make a reliable order-α^2 computation one has to take the conformal and κ ghost sectors appropriately into account and that the absence of a $D = 10$ manifest Lorentz covariance can not be so easily handled as at one loop. In particular, it may not be sufficient to impose appropriate $SO(8)$ transversality conditions on the background fields. With this respect the absence of a manifestly Lorentz covariant quantization scheme constitutes a conceptual drawback.

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Appendix A: computation of the two-gauge fields anomaly diagram

The anomaly of the second diagram in Fig. 1, by use of the anomalous vertex (75), is given by

$$\mathcal{A}^{HIJ}_{2ij}(p,q) = i\alpha \int \frac{d^Dk}{(2\pi)^D} \operatorname{tr}\left(\left(i\hat{\not{k}}\gamma^3 T^H\right)\frac{i}{\not{k}}\left(\gamma_j\frac{1+\gamma_3}{2}T^J\right)\frac{i}{\not{k}-\not{q}}\left(\gamma_i\frac{1+\gamma_3}{2}T^I\right)\frac{i}{\not{k}-\not{p}-\not{q}}\right);$$

as we need to compute this integral only in the limit for $\epsilon \to 0$, due to the presence of the hatted order-ϵ $\hat{\not{k}}$ we can set the external momenta to zero to peek the $\frac{1}{\epsilon}$-pole coming from the logarithmically divergent integral over k.

$$\begin{aligned}\mathcal{A}^{HIJ}_{2ij}(p,q) &= -\alpha \int \frac{d^Dk}{(2\pi)^D}\frac{k_m k_n k_r k_s}{(k^2)^3}\operatorname{tr}\left(T^H T^J T^I\right)\operatorname{tr}\left(\hat{\gamma}_m\gamma_3\gamma_n\gamma_j\frac{1+\gamma_3}{2}\gamma_r\gamma_i\frac{1+\gamma_3}{2}\gamma_s\right)\\ &\equiv \int \frac{d^Dk}{(k^2)^3}\mathcal{J}^{HIJ}_{ij}(k^3,\hat{k}).\end{aligned}$$

We note that the i, j indices, being external, are implicitly barred; moreover, the r index gets barred because it is constrained by two chiral projectors: $\frac{1+\gamma_3}{2}\gamma_r\frac{1-\gamma_3}{2} = \frac{1+\gamma_3}{2}\overline{\gamma}_r\frac{1-\gamma_3}{2}$. With these simplifications, we can rewrite the gamma-matrices trace as

$$\operatorname{tr}\left(\overline{\gamma}_r\gamma_s\gamma_3\hat{\gamma}_m\gamma_n\left(\frac{1-\gamma_3}{2}\gamma_i\gamma_j\right)\right);$$

now we use the fact that the integral in k can only produce symmetrized contractions of m, n, r, s indices. But since $\eta_{\bar{i}\hat{j}} = 0$ the only possibility is

$$\begin{aligned}&(\eta_{rs}\eta_{mn} + \eta_{rn}\eta_{ms})\operatorname{tr}\left(\overline{\gamma}_r\gamma_s\gamma_3\hat{\gamma}_m\gamma_n\left(\frac{1-\gamma_3}{2}\gamma_i\gamma_j\right)\right)\\ &= \operatorname{tr}\left[\left(\overline{\gamma}_r\overline{\gamma}^r\gamma_3\hat{\gamma}_m\hat{\gamma}^m + \overline{\gamma}_r\hat{\gamma}_m\gamma_3\hat{\gamma}^m\overline{\gamma}^r\right)\left(\frac{1-\gamma_3}{2}\gamma_i\gamma_j\right)\right]\end{aligned}$$

which vanishes by using the commutation properties of the γ_3 matrix with $\hat{\gamma}_m$ and $\overline{\gamma}_r$.

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